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1 Vedic Maths

1. Vedic Math-Squaring a number (Using Base Method)

(1a) Numbers Less Than Base

With the help of Vedic Maths, we can find the squares of numbers nearing the base. The numbers which can be taken as base must have 1 followed by zeroes only (i.e. the bases taken could be 100, 1000, 10000 etc.). So with the help of Base Method (Base 100), we can find the squares from 70 to 130 (We need to know the squares from 1 to 30 to get the answer).

When we find square of a number, it will have two parts, the left part and the right part. There can be any number of digits in the left part, but the right part will have number of digits equal to the number of zeros in the base i.e. if the base is 100, then there will be 2 digits on the right side and if the base is 1000, then there will be 3 digits on the right side.

Let us have a few examples to understand this concept. Let us find the square of 94 using Base Method. The base nearest to 94 is 100. It can be observed that difference between the base and the number given (94 in our case) is 6. The square of this difference is 36, which will become the right side of the answer. As the number 36 is already having two digits, so it would become the right side of our answer. Now the difference of 6 (100 – 94) is subtracted from the number given i.e. 94 – 6 = 88 and it will become the left side. Therefore the square of 94 is 8836.

Let us take another example, say 91, which is 9 less than the base. While squaring 91, the right side will be (9)2 i.e. 81. And the left side would be the number given – difference i.e. 91 – 9 = 82. So the square of 91 is 8281.

If the square of the difference is having lesser digits than required, then in order to have the required number of digits on the right side, 0's can be put with the square. e.g. If you square 98, difference is 2. The right side in this case would become 04, because 4 is a single digit number and you'll have to put a '0' before it to make it a two-digit right hand side. The left side would be 98 – 2 = 96. Hence the square of 98 is 9604.

If number of digits is more than required, then the extra digits are carried to the left side. e.g. take 88. The difference is 12 and the square of the difference is 144, which is a 3-digit number, so the 3rd extra digit 1 would be carried to the left side. And the left side is 88 – 12 = 76 + 1 (carried over) = 77. So the square of the number is

```
  7 6 - -
+ - 1 4 4
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  7 7 4 4
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You can practice the following squares to get expertise on squaring.

84______ 79______ 92______ 87______

95______ 78______ 93______ 83______

99______ 89______ 86______ 81______

(1b) Numbers More Than Base

If we want to square a number which is greater than the base, then the only difference in approach is that the in this case, the difference between the number and the base is to be added in the number instead of subtracting which we did in the last case. Let us find the square of 108. The difference is 8. The right side will be square of difference i.e. $(8)^2 = 64$. And the left side will be 108 + 8 = 116, because the number is greater than the base. So the square is 11664.

In this case also, if the number of digits on the right side is less than the required number, then we need to write '0's with it to get the right side. E.g if we want to find the square of 102, as the difference is 2, its square is 4, which is a single digit number, so a 0 would be written with it i.e. 04. Then the left side is 102 + 2 = 104. The square of 102 becomes 10404.

In case the square of the difference is a 3-digit number, then the third digit would be carried to the left side. Consider one number say 116.

The difference is 16 $\Rightarrow (16)^2 \Rightarrow 256$. Out of this 3-digit number the third digit 2 would be taken to the left side. The left side would become 116 + 16 + 2 (Carried) = 134 and the square would be

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  1 3 2 - -
+ - - 2 5 6
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  1 3 4 5 6
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Square the following numbers:

119______122______117______118______

114______107______121______109______

103______115______111______112______

2. Tables, Squares and Cubes to be remembered

As you have decided to improve your quantitative skills, but keep in mind you cannot be good at Math unless you are good at calculations. Take this as the starting point and make it the most important part of your preparation

(2a) Tables

Learn all these tables by heart and see how you improve your calculation speed.

T	T×1	T×2	T×3	T×4	T×5	T×6	T×7	T×8	T×9	T×10
12	12	24	36	48	60	72	84	96	108	120
13	13	26	39	52	65	78	91	104	117	130
14	14	28	42	56	70	84	98	112	126	140
15	15	30	45	60	75	90	105	120	135	150
16	16	32	48	64	80	96	112	128	144	160
17	17	34	51	68	85	102	119	136	153	170
18	18	36	54	72	90	108	126	144	162	180
19	19	38	57	76	95	114	133	152	171	190

(2b) Squares

Learn these squares by heart.

Z	1	2	3	4	5	6	7	8	9	10
Z^2	1	4	9	16	25	36	49	64	81	100
Z	11	12	13	14	15	16	17	18	19	20
Z^2	121	144	169	196	225	256	289	324	361	400
Z	21	22	23	24	25	26	27	28	29	30
Z^2	441	484	529	576	625	676	729	784	841	900
Z	31	32	33	34	35					
Z^2	961	1024	1089	1156	1225					

(2c) Cubes

Learn these cubes by heart

Y	1	2	3	4	5	6	7	8	9	10	11
Y^3	1	8	27	64	125	216	343	512	729	1000	1331
Y	12	13	14	15	16	17	18	19	20	21	22
Y^3	1728	2197	2744	3375	4096	4913	5832	6859	8000	9261	10648

3. Simplification

Simplification means to simplify a complicated mathematical expression to get a single answer. To understand it more clearly let us solve the following example.

Q. Solve $8 + 4 \div 2$

Sol. The correct solution is as follows
$= 8 + 4 \times 1/2 = 8 + 2 = 10$

Note: Many students will solve it as follows:
$8 + 4 \div 2 = 12 \div 2 = 6$ which is wrong. We must follow the rule of BODMAS. According to this rule multiplication should be done after division.

Rule of BODMAS	
B =	Bracket (Brackets are solved at first)
O =	Of
D =	Division
M =	Multiplication
A =	Addition
S =	Subtraction

So it means that while solving we must follow the above sequence. So in above given example we first divide 4 by 2 and add it to 8 as division (D) is before addition (A) in BODMAS.

Now we will solve a few examples.

Solved Examples:

1. $5 + 5 \times 5 \div 5$

Ans. $5 + 5 \times 5 \times \frac{1}{5} \Rightarrow 5 + 5 = 10$

2. $2 + 2 \text{ of } 4 \div 2 - \frac{17}{4}$

Ans. $2 + 8 \div 2 - \frac{17}{4}$

(Note 'of' must be solved before '÷')

$= 2 + 8 \times \frac{1}{2} - \frac{17}{4}$

$\Rightarrow 2 + 4 - \frac{17}{4} \Rightarrow \frac{6}{1} - \frac{17}{4} \quad \Rightarrow \frac{24-17}{4} = \frac{7}{4}$

3. $15 - [3 - \{2 - (5 - \overline{6+3})\}]$

Ans. This is an example where brackets are given. Brackets are solved after Bar.
The order of solving the brackets is () {} and [].
So solution of above examples is as follows.

$= 15 - [3 - \{2 - (5 - 9)\}]$

$\Rightarrow 15 - [3 - \{2 - (-4)\}]$

$= 15 - [3 - \{2 + 4\}]$

$\Rightarrow 15 - [3 - 6]$

$\Rightarrow 15 - [-3] = 15 + 3 = 18$

4. Solve $\frac{5+5\times5\div5}{5\times5\div5+5}$

Ans. $\frac{5+5\times5\div5}{5\times5\div5+5} = \frac{5+5\times5\times\frac{1}{5}}{5\times5\times\frac{1}{5}+5} = \frac{5+5}{5+5} = \frac{10}{10} = 1$

5. Solve $\frac{5\div5 \text{ of } 5+5}{5\div5\times5+5}$

Ans.

$\frac{5\div5 \text{ of } 5+5}{5\div5\times5+5} \Rightarrow \frac{5\div25+5}{5\times\frac{1}{5}\times5+5}$

$= \frac{5\times\frac{1}{25}+5}{5+5} = \frac{\frac{1}{5}+5}{10} \Rightarrow \frac{26}{5}\times\frac{1}{10} = \frac{13}{25}$

$= \frac{5\times\frac{1}{25}+5}{5+5} = \frac{\frac{1}{5}+5}{10} \Rightarrow \frac{26}{5}\times\frac{1}{10} = \frac{13}{25}$

2 Number System

Definitions & Divisibility Rules & Integers

1. Definitions

(1a) Natural Numbers:

The numbers starting from 1, 2, 3, 4, ... and so on that comes naturally when we start counting are known as Natural numbers. They do not contain any decimal or fraction. Natural numbers are also called positive integers.

(1b) Whole Numbers:

If we consider zero (0) as the predecessor of 1 and add it to the collection of natural numbers, we call them whole numbers. Whole numbers are also called non-negative integers.

(1c) Prime and Composite Numbers:

There are natural numbers, having exactly two distinct factors, 1 and the number itself.

Such numbers are 2, 3, 5, 7, 11 etc. These numbers are known as prime numbers.

The numbers other than 1, whose only factors are 1 and the number itself are called Prime numbers.

The numbers other than prime which have more than two factors like 4, 6, 8, 9, 10 etc are known as composite numbers. *The numbers having more than two factors are called Composite numbers.*

1 (one) is neither a prime nor a composite number.

2 (two) is the smallest prime number which is even. Every prime number except 2 is odd.

There are total of 15 prime numbers from 1 to 50 and 25 prime numbers from 1 to 100 which are written below:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

(1d) Even and Odd Numbers:

The numbers which are divisible by 2 are known as even numbers. For example, 0, 2, 4, 6, 8, ... and so on. The rest of the numbers which are not divisible by 2 are known as odd numbers. For Example, 1, 3, 5, 7, 9, ... and so on.

(1e) Factors and Multiples:

Number 6 can be written as: $6 = 1 \times 6 = 2 \times 3 = 3 \times 2 = 6 \times 1$ and the numbers 1, 2, 3 and 6 are exact divisors of 6. These divisors are called factors of 6.

Any factor of a number is an exact divisor of that number and each factor of the number is less than or equal to that number.

Again, we write a number 15 as $15 = 3 \times 5$, we say 3 and 5 are factors of 15. We also say that 15 is a multiple of 3 and 5.

The representation $33 = 3 \times 11$ shows that 3 and 11 are factors of 33, whereas 33 is a multiple of 3 and 11.

A number is divisible by each of its factors.

(1f) Properties of Factors and Multiples:

1. 1 (one) is a factor of every number. For example, $16 = 1 \times 16$, $3 = 1 \times 3$, etc.
2. Each number is a factor of itself. For example, one may write 7 as 7×1, 13 as 13×1, etc.
3. Each factor of a number is an exact divisor of that number. The factors of 12 are 1, 2, 3, 4, 6, 12 only.
4. Each number is a multiple of itself. For example, $8 = 8 \times 1$, $13 = 13 \times 1$ and so on.

(1g) Common Factors and Common Multiples:

The numbers 15 and 20 have common factors as 1 and 5 and common multiples as 60, 120, 180 etc. If two numbers have only 1 as a common factor, then they are known as co-prime numbers. Thus, 5 and 12 are co-prime numbers.

2. Divisibility Rules

Divisibility by 10: If a number has 0 in the units' place, then it is divisible by 10. For example, 320 is divisible by 10.

Divisibility by 5: A number that has either 0 or 5 in its units' place is divisible by 5. For example, 125 and 270 both are divisible by 5.

Divisibility by 2: A number is divisible by 2 if it has its units' place as 0, 2, 4, 6 or 8. For example, 672, 44, 538, 30, 136 etc. are all divisible by 2.

Divisibility by 3: If the sum of the digits of the number is a multiple of 3, then the number is divisible by 3. For example, sum of digits of 378 = 3 + 7 + 8 = 18 which is divisible by 3. Hence, 378 is divisible by 3.

Divisibility by 6: If a number is divisible by 2 and 3 both, then it is divisible by 6 as well. For example, 72 is divisible by 2 and 3 both so it is divisible by 6 also.

Divisibility by 4: A number is divisible by 4 if the number formed by its last two digits (i.e., units and tens) is divisible by 4. For example, last two digits of 1348 are 48 which is divisible by 4, so 1348 is divisible by 4.

Divisibility by 8: A number is divisible by 8, if the number formed by the last three digits is divisible by 8. For example, last three digits of 42328 are 328 which is divisible by 8, so 42328 is divisible by 8.

Divisibility by 9: If the sum of the digits of a number is divisible by 9, then the number itself is divisible by 9. For example, sum of digits of 13068 = 1 + 3 + 0 + 6 + 8 = 18, which is divisible by 9, so 13068 is divisible by 9.

Divisibility by 11: If difference between the sum of the digits at odd places and the sum of the digits at even places of a number is calculated, which is found to be either 0 or divisible by 11, then the number is divisible by 11. For example, the sum of the digits at odd places and the sum of the digits at even places of the number 26983 is 2 + 9 + 3 = 14 and 6 + 8 = 14 respectively. Now difference is 14 – 14 = 0, so 26983 is divisible by 11.

Prime Factorization:

When a number is expressed as a product of its factors, we say that the number has been factorized. Thus, when we write $48 = 6 \times 8$, we say that 48 has been factorized. This is one of the factorizations of 48. The others are: $48 = 2 \times 24 = 3 \times 16 = 4 \times 12 = 6 \times 8 = 2 \times 2 \times 2 \times 2 \times 3$

In all the above factorizations of 48, we ultimately arrive at only one factorization $2 \times 2 \times 2 \times 3$. In this factorization the only factors 2 and 3 are prime numbers. Such a factorization of a number is known as prime factorization.

3. Integers

Natural numbers are known as positive integers and whole numbers are known as non-negative integers. If we take natural numbers and put negative sign ahead of them, they are known as negative integers i.e., -1, -2, -3, -4, -5, ... and so on. Integers are combination of whole numbers as well as negative integers together as ... -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ... and so on

4. Rational Numbers

A rational number, which is derived from the term ratio, is defined as the ratio of two integers p and q and can be expressed in the form $\frac{p}{q}$ where $q \neq 0$. Thus, $\frac{4}{5}$ is a rational number.

Here, p = 4 and q = 5. $\frac{-3}{4}$ is also a rational number where p = – 3 and q = 4 are integers.

Other fractions like $\frac{5}{7}, \frac{4}{9}, 2\frac{3}{5}$ etc. are all rational numbers. All fractions are rational numbers.

Decimal & Fractions

A fraction means a part of a total or of a sum.

For example, 7/8 is a fraction. We read it as "seven-eighths". Here "8" stands for the number of equal parts into which the whole has been divided. And here "7" stands for the number of equal parts which have been taken out. So, 7 things have been divided into 8 equal parts such that each part is 7/8.

Here 7 is called the numerator and 8 is called the denominator.

1. Proper Fractions

All those fractions which lies between 0 and 1 are known as proper fractions. In a proper fraction, the numerator is always less than the denominator.

2. Improper and Mixed Fractions

The fraction in which the numerator is greater than the denominator is called as improper fraction. Thus, fractions like $\frac{5}{3}, \frac{9}{5}, \frac{13}{7}$ ..., are all improper fractions.

Fractions such as $2\frac{3}{4}$ and $3\frac{1}{3}$ are known as mixed fractions. A mixed fraction is a combination of a whole number and a proper fraction.

We can express a mixed fraction as an improper fraction as $\frac{(\text{Whole} \times \text{Deno min ator}) + \text{Numerator}}{\text{Deno min ator}}$

3. Like & Unlike Fractions

Fractions with same denominators are known as like fractions. Thus, $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{8}{15}$....... are all like fractions.

Fractions having different denominators are known as unlike fractions. For example. $\frac{7}{27}$ and $\frac{7}{28}$.

(3a) Comparing like Fractions

The two like fractions can be simply compared by checking the numerators.

Let us compare two like fractions: $\frac{3}{8}$ and $\frac{5}{8}$.

Since, $5 > 3$, hence $\frac{5}{8} > \frac{3}{8}$.

So, we may deuce that for two fractions with the same denominator, the fraction with the greater numerator is greater. Between $\frac{4}{5}$ and $\frac{3}{5}$, $\frac{4}{5}$ is greater. Between $\frac{11}{20}$ and $\frac{13}{20}$, $\frac{13}{20}$ is greater and so on.

(3b) Comparing unlike Fractions

The unlike fractions can be compared simply by cross multiplying the denominators with the numerators. Bigger is the value, bigger is the fraction.

Consider two unlike fractions $\frac{3}{5}$ and $\frac{5}{6}$.

Cross multiply the denominators with the numerators.
$3 \times 5 = 15$ and $5 \times 5 = 25$,

Since, $15 < 25$, therefore $\frac{3}{5} < \frac{5}{6}$. $3/5 < 5/6$

4. Fractions as Decimals

Let us try to find decimal representation of

(a) $\frac{11}{5}$ (b) $\frac{1}{2}$

(a) We know that

$$\frac{11}{5} = \frac{22}{10} = \frac{(20+2)}{10} = \frac{20}{10} + \frac{2}{10} = 2 + \frac{2}{10} = 2.2.$$

Therefore, $\frac{22}{10} = 2.2$

(b) Likewise, $\frac{1}{2} = \frac{5}{10} = 0.5.$

5. Decimals as Fractions

To write decimal value into fraction, divide the number with as many multiples of 10 as there are after decimal value. For example, $1.2 = 1 + \frac{2}{10} = \frac{10}{10} + \frac{2}{10} = \frac{12}{10}$

6. Comparing Decimals

To compare decimal values, we simply observe the place value of that decimal, higher is the place value, higher is number. Let's say we have to compare 32.55 and 32.5. In this we may observe that 32.5 is common in both the decimal numbers, but in 1[st] number 32.55, we have extra place value of 0.05. Hence, $32.55 > 32.5$

Indices & Exponents

1. Exponents and Powers

The distance between the earth and sun is approximately 150,000,000 km, which can be more conveniently written as 1.5×10^8 km using exponents. We read 10^8 as 10 raised to the power 8, where 10 is the base and 8 is the exponent.

Likewise, $3^5 = 3 \times 3 \times 3 \times 3 \times 3$ and
$3^m = 3 \times 3 \times 3 \times 3 \times \ldots$ (m times)

(1a) Powers with Negative Exponents

We know that, $10^2 = 10 \times 10 = 100$,

$10^1 = 10$ i.e., $\frac{100}{10}$,

$10^0 = 1$ i.e., $\frac{100}{10 \times 10}$

$10^{-1} = ?$

Continuing the above pattern we get, $10^{-1} = \frac{1}{10}$ as the exponent decreases by 1, the value becomes one-tenth of the previous value.

Similarly, $10^{-2} = \frac{1}{10} \div 10 = \frac{1}{10} \times \frac{1}{10} = \frac{1}{100} = \frac{1}{10^2}$

We have, $10^{-2} = \frac{1}{10^2}$ or $10^2 = \frac{1}{10^{-2}}$

$10^{-3} = \frac{1}{10^3}$ or $10^3 = \frac{1}{10^{-3}}$

$5^{-2} = \frac{1}{5^2}$ or $5^2 = \frac{1}{5^{-2}}$

In general, we can say that for any non-zero integer a, $a^{-m} = \frac{1}{a^m}$, where m is a positive integer. a^{-m} is the multiplicative inverse of a^m.

(1b) Laws of Exponents

For non-zero integers a and b and m and n are integers.

1. $a^m \times a^n = a^{m+n}$,
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $a^m \times b^m = (ab)^m$
5. $\frac{a^m}{b^m} = \left(\frac{a}{b}\right)^m$
6. $a^0 = 1$

Square Root & Cube Root

1. Squares and Square Roots

Consider 25, which can be expressed as $5 \times 5 = 5^2$. Likewise, 81 can be expressed as $9 \times 9 = 9^2$ etc. All such numbers that can be expressed as the product of the number with itself are known as square numbers. In general, if a natural number n can be expressed as x^2, where x is also a natural number, then n is a square number.

The following table shows the squares of numbers from 1 to 20.

Number	Square	Number	Square
1	1	11	121
2	4	12	144
3	9	13	169
4	16	14	196
5	25	15	225
6	36	16	256
7	49	17	289
8	64	18	324
9	81	19	361
10	100	20	400

(1a) Properties of Square Numbers:

1. A square number always ends with either 0, 1, 4, 5, 6 or 9 at unit's place. And never ends with 2, 3, 7 or 8 at unit's place.
2. If a number has 1 or 9 in the unit's place, then it's square ends in 1. If unit's place is 2 or 8, square ends with 4. If unit's place ends with 3 or 7, square ends with 9. If unit's place ends with 4 or 6, square ends with 6. In case of unit's place is 0 or 5, it ends with 0 or 5 respectively.

(1b) Square Roots:

Consider a number 225 is given and we have to find that this number is square of which number.

Now, $225 = 3 \times 3 \times 5 \times 5 = 15 \times 15 = 15^2$. So, we can say that 15 is the square root of number 225.

Finding square root through prime factorization

Prime factorization of a Number	Prime factorization of its Square
$10 = 2 \times 5$	$100 = 2 \times 2 \times 5 \times 5$
$18 = 2 \times 3 \times 3$	$324 = 2 \times 2 \times 3 \times 3 \times 3 \times 3$
$21 = 3 \times 7$	$441 = 3 \times 3 \times 7 \times 7$

We can observe that for each prime factor in the given number, we have twice the number of times the same factor occurs in the prime factorization of the square of that number.

Let's try finding the square root of a given number, say 1764.

The prime factorization of $1764 = 2 \times 2 \times 3 \times 3 \times 7 \times 7$
$= 2^2 \times 3^2 \times 7^2 = (2 \times 3 \times 7)^2 = 42^2$

So, $\sqrt{1764} = 2 \times 3 \times 7 = 42$.

For examples:

1. Find the square root of 8100.

Ans. Write $8100 = 2 \times 2 \times 3 \times 3 \times 3 \times 3 \times 5 \times 5$.
Therefore $\sqrt{8100} = 2 \times 3 \times 3 \times 5 = 90$

2. Is 180 a perfect square?

Ans. We have $180 = 2 \times 2 \times 3 \times 3 \times 5$. The prime factor 5 does not occur in pairs. Therefore, 180 is not a perfect square. That 180 is not a perfect square can also be seen from the fact that it has only one zero.

2. Cubes

Consider the numbers 1, 8, 27, 64 and so on. These numbers are known as **perfect cubes or cube numbers** as each of them is obtained when a number is multiplied by itself three times.

By observation, $1 = 1 \times 1 \times 1 = 1^3$, $8 = 2 \times 2 \times 2 = 2^3$, $27 = 3 \times 3 \times 3 = 3^3$ etc.

Consider, $128 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7$ is not a perfect cube number.

Following table shows the cubes of numbers from 1 to 20.

Number	Cube	Number	Cube
1	1	11	1331
2	8	12	1728
3	27	13	2197
4	64	14	2744
5	125	15	3375
6	216	16	4096
7	343	17	4913
8	512	18	5832
9	729	19	6859
10	1000	20	8000

(2a) Smallest multiple that is a perfect cube:

1. Is 1080 a perfect cube? If not, then find the smallest natural number, by which should 1080 be divided so that the quotient is a perfect cube?

Ans. $1080 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5$. The prime factor 5 does not appear in a group of three. So, 1080 is not a perfect cube.

In the factorization of 1080, 5 appears only one time. If we divide the number by 5, then the prime factorization of the quotient will not contain 5. So, $1080 \div 5 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$. Hence the smallest number by which 1080 should be divided to make it a perfect cube is 5. The perfect cube in that case is 216.

(2b) Cube Roots:

Consider a number 3375 is given and we have to find that this number is cube of which number.
Now, $3375 = 3 \times 3 \times 3 \times 5 \times 5 \times 5 = 15 \times 15 \times 15 = 15^3$.
So, we can say that 15 is the cube root of number 3375.

(2c) Cube root through prime factorization method:
Consider 9261. We find its cube root by prime factorization:
$9261 = 3 \times 3 \times 3 \times 7 \times 7 \times 7 = 3^3 \times 7^3 = (3 \times 7)^3$
Therefore, cube root of $9261 = 3 \times 7 = 21$

HCF & LCM

1. HCF & LCM

Greatest Common Divisor (GCD) / Highest Common Factor (HCF):

Highest Common Factor of two or more numbers is that greatest number, which divides each of those numbers an exact number of times. e.g. HCF of 24 and 36 is 12.

How to find the HCF of two or more numbers?

a) Express the two numbers as the product of prime numbers separately.

b) Take the product of prime numbers common to both numbers.

For examples:

1. What is the HCF of 12, 54 and 36?

Ans. $12 = 2 \times 2 \times 3$, $54 = 2 \times 3 \times 3 \times 3$, $36 = 2 \times 2 \times 3 \times 3$. The greatest common factor of 12, 54 and 36 is 2×3. Thus, HCF of 12, 54, 36 is 6

Least Common Multiple - LCM

The least common multiple (LCM) of two or more numbers is the smallest of the numbers, which is exactly divisible by each one of them.

e.g. Consider two numbers 18 and 15
Multiples of 18 are: 18, 36, 54, 72, 90, 108, 126, 144, 162, 180....
Multiples of 15 are: 15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, ...
The common multiples of both 18 and 15 are 90, 180, ...
The least common multiple is 90.

Approach of finding the LCM of two or more numbers:

LCM of two or more numbers can be calculated by the product of the factors of the two numbers after eliminating repetition of the common factors.

In the above Ex., the common factor for 9 and 15 is 3. Therefore, the LCM will be $3 \times 3 \times 5 = 45$.

Alternatively, LCM is the product of all prime factors of the given numbers, the common factors among them being in their highest degree. e.g., The LCM of $5x^2y^3z^5$ and $3xy^2z^7$ will be $5 \times 3 \times x^2y^3z^7 = 15x^2y^3z^7$, where x, y and z are the prime factors.

2. Find the LCM of 16 and 24

Ans. We know that common multiples of 16 and 24 are 48, 96, 144 etc. The lowest of these is 48. Now we go through another method for finding the LCM of two numbers.
We can do prime factorization of 16 and 24 which is: $16 = 2 \times 2 \times 2 \times 2$; $24 = 2 \times 2 \times 2 \times 3$
In these prime factorizations, we can observe that in the number 16, prime factor 2 occurs maximum four times. Similarly, in 24, the factor 3 occurs maximum one time. The LCM of the two numbers is calculated as the product of the prime factors which are counted the most number of times they occur in any of the numbers. So, in this case LCM $= 2 \times 2 \times 2 \times 2 \times 3 = 48$.

3. ind the LCM of 48 and 180.

Ans. The prime factorizations of 48 and 180 are: $48 = 2 \times 2 \times 2 \times 2 \times 3$; $180 = 2 \times 2 \times 3 \times 3 \times 5$. In these prime factorizations, we can observe that in the number 48, prime factor 2 occurs maximum four times. Similarly, in the number 180, the prime factor 3 occurs two times and prime factor 5 occurs only once. Thus, LCM $= (2 \times 2 \times 2 \times 2) \times (3 \times 3) \times 5 = 720$

4. Find the LCM of 20, 24 and 15.

Ans. The prime factorizations of 20, 24 and 15 are;
$20 = 2 \times 2 \times 5$
$24 = 2 \times 2 \times 2 \times 3$
$15 = 3 \times 5$
The prime factor 2 appears maximum three times in the prime factorization of 24, the prime factor 3 occurs only one time in the prime factorization of 24

and 15, the prime factor 5 appears exactly once in the prime factorizations of 20 and 15, so we take it only once.

Therefore, required LCM = $(2 \times 2 \times 2) \times 3 \times 5 = 120$.

2. LCM and HCF of Fractions

$$\text{LCM of fractions} = \frac{\text{LCM of numerators}}{\text{HCF of denominators}};$$

$$\text{e.g. LCM of } \frac{3}{4} \text{ and } \frac{1}{2} = \frac{3\,(\text{LCM of numerators})}{2\,(\text{HCF of denominators})}$$

$$\text{HCF of fractions} = \frac{\text{HCF of numerators}}{\text{LCM of denominators}}$$

$$\text{e.g. HCF of } \frac{3}{4} \text{ and } \frac{1}{2} = \frac{1(\text{HCF of numerators})}{4(\text{LCM of denominators})}$$

Note that the product of the two fractions is always equal to the product of LCM and HCF of the two fractions.

The product of the two fractions = $\frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$.

The product of the LCM and HCF = $\frac{3}{2} \times \frac{1}{4} = \frac{3}{8}$.

3. Important Points in LCM and HCF

- ✓ In case of HCF, if some remainders are given, then we need to subtract those remainders from the numbers given and then calculate their HCF.
- ✓ In case of LCM, if a single remainder is given, then we need to calculate the LCM first and then that the single reminder is added in that.
- ✓ In case of LCM, if for different numbers, different remainders are given, then the difference between the number and its respective remainder will be equal. In that case, we need to calculate the LCM first, then that common difference between the number and its respective remainder is subtracted from that.
- ✓ Sometimes in case of HCF questions, the same remainder is required and the remainder is not given.
- ✓ If the question asks about the maximum or greatest, then most likely, it's a question of HCF. Also if the question is related to distribution or classification into groups, then in all likelihood, it's a question of HCF only.
- ✓ If the question asks about the smallest or minimum, then most likely, it's a question of LCM. Also, if the word 'together' or 'simultaneous' is used in the question, then in all likelihood, it is a question of LCM only.

3. Average

1. Definitions

Average also known as arithmetic mean, is basically the ratio of sum of the terms to the number of terms. Numerically the average can be found as:

$$\text{Average} = \frac{\text{Sum of Items}}{\text{Number of Items}}$$

For Examples:

1. Find the average of 15, 21, 33, 37, 39.

Ans. $\text{Average} = \frac{(15+21+33+37+39)}{5} = \frac{145}{5} = 29$

Note: When the difference between all the terms is same, then average is equal to $\frac{(n+1)}{2\text{th}}$ term if 'n' is odd and equal to the average of $\frac{n}{2\text{th}}$ and $\left(\frac{n}{2}+1\right)$th term when 'n' is even, where n is the total number of terms.

2. Find average of 53, 57, 61, 65, 69.

Ans. Here difference between the terms is same and number of terms are 5. So average will be $\frac{(5+1)}{2}$ = 3rd term which is 61. So average is 61

2. Important Facts

If each item in the list is increased or decreased by a certain quantity x, then the average / mean will also increase or decrease by the same quantity x.

If each item in the list is multiplied or divided by a certain quantity x, then the average / mean will also get multiplied or divided by the same quantity x.

If a number is added to half of the quantities and the same number is subtracted from other half of the quantities then the average value remains same as original average.

3. Weighted Mean/Average

The weighted arithmetic mean is denoted by:

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots\ldots + w_nx_n}{w_1 + w_2 + w_3 + \ldots. + w_n}$$

Where $x_1, x_2, x_3, \ldots, x_n$ are averages and $w_1, w_2, w_3, .., w_n$ are their respective weights.

4. Average Speed

If a man covers some journey from A to B at u km/hr and returns to A from B with uniform speed of v km/hr, then the average speed of the complete journey is $\frac{2uv}{u+v}$ km/hr

4 Algebra

Algebraic Expression

1. Algebraic Expression

There are different kinds of algebraic expression like *monomial, binomial, trinomial* etc. In *latin* the **mono** means **one**, **bi** means **two**, **tri** means **three** and so on. So the roots of these words help us to understand these terms. Let us discuss each of them one by one.

(i) **Monomial:** The *algebraic expression having one term* is called **monomial** e.g. **3b, 6c, $8a^2b$** etc.

(ii) **Binomial:** The **algebraic expression having two terms** is called **binomial** e.g. **$3b + 5c^2$, $6c^3 + 8e$** etc.

(iii) **Trinomial:** The algebraic expression having **three** terms is called **trinomial** e.g. **$3x + y + 2z$, $5a + 3b^2 + 9c$** etc.

(iv) **Polynomial:** In general an **algebraic expression having one or more terms** is called **polynomial.**

2. Algebraic Expressions and Identities

(2a) Expressions

We are already familiar with what algebraic expressions (or simply expressions) are.

*Example*s of expressions are:
$x + 1, 2y - 2, 3x^2, 2xy + 7$ etc.

We can form many more expressions as we know that expressions are formed from variables and constants. The expression $3y - 6$ is formed from the variable y and constants 3 and 6.

The expression $2xy + 11$ is formed from variables x and y and constants 2 and 11. We know that in the expression $2y - 5$, the value of y could be anything like 1, 2 , 3.. etc.

(2b) Identity

An identity is equality, which is true for all values of the variables $(a + 2)(a + 3) = a^2 + 5a + 6$ *is an identity* in the equality. On the other hand, an equation is true only for certain values of its variables. An equation is not an identity

For *example*, consider the equation $a^2 + 5a + 6 = 56$
It is true for a = 5, as seen above, but it is not true for a = 10 or for a =1 etc

The following are the standard identities:

$(a + b)^2 = a^2 + 2ab + b^2$ ...(i)
$(a - b)^2 = a^2 - 2ab + b^2$ ...(ii)
$(a + b)(a - b) = a^2 - b^2$ ...(iii)

For examples:

1. Using the Identity
Find (i) $(2x + y)^2$ (ii) 106^2

Ans. (i) $(2x + y)^2 = (2x)^2 + 2(2x)(y) + (y)^2$
$= 4x^2 + 4xy + y^2$
(ii) $(106)^2 = (100 + 6)^2$
$= 100^2 + 2 \times 100 \times 6 + 6^2$ (Using Identity I)
$= 10000 + 1200 + 36 = 11236$

2. If $x + \frac{1}{x} = 3$, what is the value of $x^2 + \frac{1}{x^2}$?

Ans. $x + \frac{1}{x} = 3$ (squaring both sides)

$x^2 + \frac{1}{x^2} + 2 \times x \times \frac{1}{x} = 9$

$\Rightarrow x^2 + \frac{1}{x^2} = 9 - 2 = 7$

Some more important identities

i. $(a + b)^2 - (a - b)^2 = 4ab$

ii. $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

iii. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

iv. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

v. $(a^3 + b^3) = (a + b)(a^2 + b^2 - ab)$

vi $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$

vii $(a + b + c)^2 = [a^2 + b^2 + c^2 + 2(ab + bc + ca)]$

viii. $(a + b + c + d)^2 = [a^2 + b^2 + c^2 + d^2 + 2a(b + c + d) + 2b(c + d) + 2cd] = [a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd]$

ix $(a^3 + b^3 + c^3 - 3abc) = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ *If* $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

x $(x + a)(x + b) = x^2 + (a + b)x + ab$

3. Find the value of $m^3 + n^3$, given mn = 200, (m + n) = 30.

Ans. m + n = 30 (cubing both sides)
$m^3 + n^3 + 3mn(m + n) = 27000$. (Replacing the variables by two values given)
$\Rightarrow m^3 + n^3 = 27000 - 18000 = 9000.$

Linear Equations in one Variable

(a) An algebraic equation is an equality which involves variables. It always has an equality sign between the expression written on the left (called as LHS) and the expression written on the right (called as RHS)

(b) In an equation, the values of expressions written on LHS and RHS are always equal. This is true only for a certain values of the variable which are known as the solutions of the equation.

(c) If asked to find the solution of an equation, we will assume that the two sides of the equation are balanced. We need to perform the same mathematical operation on both sides of the equation, so that the balance is maintained. Then we can obtain the solution.

Let us revive the approaches of solving equations with some *examples.*

1. Find the solution of $3x - 5 = 10$

Ans. *Step 1:* Add 5 to both sides.

$3x - 5 + 5 = 10 + 5$

(The balance is not disturbed) or $3x = 15$

Step 2: Next divide both sides by 3.

$\frac{3x}{3} = \frac{15}{3}$

$\Rightarrow x = 5$

2. Sum of two numbers is 38. One of the numbers is 8 more than the other. What are the numbers?

Ans. As we don't know either of the two numbers, we have to find them, given two conditions.

(i) One of the numbers is 8 more than the other.

(ii) Their sum is 38

If we take the smaller number to be x, then the larger number is 8 more than x, i.e., $x + 8$.

The second condition tells us that the sum of these two numbers x and $x + 8$ is 38.

This means that $x + (x + 8)$

$= 38$ or $2x + 8 = 38$,

$2x = 38 - 8$ or $2x = 30$,

Dividing both sides by 2.

$x = 15$.

This is one number.

The other number is $x + 8 = 15 + 8 = 23$.

The desired numbers are 15 and 23

1. Linear Equations with two unknowns

The following two methods are normally adopted to solve linear equations in two variables:

3. Solve to find value of x and y.

$3x + 2y = 11 \qquad x - y = 2$

Ans. The following two equations can be solved for x and y.

$$3x + 2y = 11 \qquad \ldots(1)$$
$$x - y = 2 \qquad \ldots(2)$$

In equation (2), $x = 2 + y$.

Substitute $(2 + y)$ in equation (1) for x:

$3(2 + y) + 2y = 11$

$6 + 3y + 2y = 11 \Rightarrow 6 + 5y = 11$

$\Rightarrow \quad 5y = 5 \Rightarrow y = 1$.

If $y = 1$, then $x = 2 + 1 = 3$.

4. Solve to find value of x and y.

$3x + 2y = 11 \qquad x - y = 2$

Ans. The following two equations can be solved for x and y.

$$3x + 2y = 11 \qquad \ldots(1)$$
$$x - y = 2 \qquad \ldots(2)$$

multiply equation (2) by 2 we get the following:

$$3x + 2y = 11 \qquad \ldots(1)$$
$$2x - 2y = 4 \qquad \ldots(3)$$

Adding these two equations,

we get $\quad 5x = 15 \Rightarrow x = 3$

Now putting the value of x in equation (1) we get

$3 \times 3 + 2y = 11$

$9 + 2y = 11$

$2y = 11 - 9 = 2$

$2y = 2 \Rightarrow y = 1$

Quadratic Equations

The standard way of writing a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. All quadratic equations are compared with standard form to find the value of a, b, and c.

For example, In $x^2 - 7x + 12 = 0$ if we compare it with $ax^2 + bx + c = 0$ $a = 1$, $b = -7$ and $c = 6$

There are two ways of solving the quadratic

(i) Solving Quadratic Equations by Factoring

(ii) Solving Equations by using the Quadratic Formula

We will discuss each one by one

1. Solving Quadratic Equations by Factoring

The standard form for a quadratic equation is $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$;

For example, $x^2 + 6x + 5 = 0$, $3x^2 - 2x = 0$, and $x^2 + 4 = 0$.

Some equations can be solved by factoring.

For this, firstly add or subtract expressions so that all the expressions are on one side of the equation, with 0 on the other side. Then try to factor the non-zero side into a product of expressions. Then individually, each factor can be set equal to 0, and accordingly the roots can be found.

As an *example*, consider the equation

$x^2 - 11x = -30$: $x^2 - 11x + 30 = 0$ (taking all terms on one side and putting the expression equal to zero)

Now try to break b into two parts, such that the sum of those two parts = 'b' and the product is equal to the product of 'a' and 'c'.

$$x^2 - 6x - 5x + 30 = 0$$

$\Rightarrow$ $x(x-6) - 5(x-6) = 0$

$\Rightarrow$ $(x-5)(x-6) = 0$

Putting these separately equal to 0

$\Rightarrow$ $x - 5 = 0, x = 5$ and $x - 6 = 0, x = 6$

Thus the solutions of the equation are 5 and 6

The solutions of an equation are also called the roots of the equation.

A quadratic equation has at most two real roots and may have just one or even no real root.

For *example*, the equation $x^2 - 6x + 9 = 0$ can be expressed as $(x-3)^2 = 0$, or $(x-3)(x-3) = 0$; thus the only root is 3. The equation $x^2 + 4 = 0$ has no real root; since the square of any real number is greater than or equal to zero, $x^2 + 4$ must be greater than zero.

An expression of the form $a^2 - b^2$ can be factored as $(a-b)(a+b)$.

For *example*, the quadratic equation $4x^2 - 9 = 0$ can be solved as follows.

$(2x-3)(2x+3) = 0$

$\Rightarrow 2x - 3 = 0$ or $2x + 3 = 0$

$\Rightarrow x = 3/2$ or $x = -3/2$

2. Solving Equations by using the Quadratic Formula

If a quadratic expression is not easily factored, then its roots can always be found using the quadratic formula:

If $ax^2 + bx + c = 0$ $(a \neq 0)$, then the roots are

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } x = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

These are two distinct real numbers if $b^2 - 4ac > 0$.

If $b^2 - 4ac = 0$; then these two expressions for x are equal to $-b/2a$ and the equation has only one root.

If $(b^2 - 4ac) < 0$, then $\sqrt{b^2-4ac}$ is not a real number and the equation has no real roots.

To solve the quadratic equation $x^2 - 6x + 4 = 0$ using the above formula, note that $a = 1$, $b = -6$, and $c = 4$, and hence the roots are

$$x = \frac{6 \pm \sqrt{6^2 - 4\times1\times4}}{2\times1} = \frac{6\pm\sqrt{20}}{2a}$$

$= 3+\sqrt{5}$ & $3-\sqrt{5}$ $b^2 - 4ac$ is called the discriminant and is denoted by the symbol Δ or is represented by the letter *D*. Following are some of the important points relating to the discriminant and its relation with the nature of the roots.

- If $\Delta > 0$, then both the roots will be real and unequal and the value of roots will be $\dfrac{-b\pm\sqrt{\Delta}}{2a}$. If Δ is a perfect square, then roots are rational otherwise they are irrational.
- If $\Delta = 0$, then roots are real, equal and rational. In this case the value of roots will be $\dfrac{-b}{2a}$.
- If $\Delta < 0$, then roots will be imaginary, unequal and conjugates of each other.
- If α and β are the roots of the equation $ax^2 + bx + c = 0$, then sum of the roots i.e. $\alpha + \beta = \dfrac{-b}{a}$.
- If α and β are the roots of the equation $ax^2 + bx + c = 0$, then product of the roots i.e. $\alpha\beta = \dfrac{c}{a}$.
- If α and β, the two roots of a quadratic equation are given, then the equation will be

  $$x^2 - (\alpha+\beta)x + \alpha\beta = 0.$$

The equation is x^2 – (sum of roots) x + product of roots = 0

These were some very important points relating to the quadratic equations. The following are some properties regarding the roots of the equation.

- If in the equation $b = 0$, then roots are equal in magnitude, but opposite in sign.
- If $a = c$, then roots are reciprocal of each other.
- If $c = 0$, then one of the roots will be zero.
- If one root of a quadratic equation be a complex number, the other root must be its conjugate complex number i.e.

  $$\alpha = j + \sqrt{-k}, \text{ then } \beta = j - \sqrt{-k}.$$

For examples:

1. Solve: $x^2 + 3x + 2 = 0$

Ans. We have $x^2 + 3x + 2 = 0$

$\Rightarrow (x+2)(x+1) = 0$

$\Rightarrow$ $x + 2 = 0$ or $x + 1 = 0$

$\Rightarrow$ $x = -2$ or $x = -1$

2. If one root of the quadratic equation $x^2 - 25x + 5z = 0$ is four times the other, find the value of z.

Ans. Here in this equation $a = 1$, $b = -25$ and $c = 5z$.

From the formula, sum of the roots = 25 (i)

If one root is α, the other root is 4α and the sum of roots will be 5α

$$5\alpha = 25 \Rightarrow \alpha = 5.$$

Other root will be $4 \times 5 = 20$

Now the product of the roots will be $= \frac{c}{a} = 5z$

Product of roots $\alpha \times 4\alpha = 4\alpha^2$
$\Rightarrow 4(5)^2$
$\Rightarrow$ Now $= 5z = 100$
$\Rightarrow \quad z = 20$

3. If $x^2 - 16x + m = 0$ have equal roots, find the value of m.

Ans. Because it has equal roots, discriminant should be equal to zero.
$(-16)^2 - 4m = 0$
$\Rightarrow \quad 256 = 4m$
$\Rightarrow \quad m = 256/4 = 64$

Progressions

In this we will discuss the following
A. Arithmetic progression
B. Geometric Progression

Let us first discuss **Arithmetic Progression**

1. Arithmetic Progression

An arithmetic progression is a sequence of numbers in which each term is derived from the preceding term by adding or subtracting a fixed number called the common difference.
Example, the sequence 9, 6, 3, 0, -3,.... is an arithmetic progression with –3 as the common difference.
The progression –3, 0, 3, 6, 9 is an Arithmetic Progression (AP) with 3 as the common difference.
Now we will discuss important formulae of AP

(1a) General term of AP

The general form of an Arithmetic Progression is $a, a + d, a + 2d, a + 3d$ and so on.
Thus nth term of an AP series is $\mathbf{T_n = a + (n - 1) d}$
where $T_n = n^{th}$ term, a = first term and d = common difference

1. Find the 9^{th} term of series 3, 7, 11, 15....

Ans. As common difference is same so it is an AP.
Here a = 3, d = 4 and n = 9
So applying the formula
$Tn = a + (n - 1) d$
We get $T_9 = 3 + (9 - 1) \times 4$
$= 3 + 8 \times 4 = 3 + 32 = 35$

(1b) Sum of n terms of an AP

There are two formulae for sum of n terms of an AP
Sum of first n terms of an AP is

$\frac{n}{2}\left[2a+(n-1)d\right]$

This formula is used when last term is not given

2. Find the sum of first ten terms of series 3, 7, 11, 15...

Ans. As common difference is same so it is an AP and also last term is not given.
So we will apply the formula

(a) $S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

Here a = 3, d = 4 and n = 9

$S_{10} = \frac{10}{2}\left[2\times3+(10-1)\times4\right]$

$= 5 [6 + 9 \times 4] = 5 [6 + 36] = 5 \times 42 = 210$ (b)

Sum of first n terms of an AP is $S_n = \frac{n}{2}\left[a+1\right]$

This formula is used when last term is also given

3. Find the sum of the series 4, 8, 12, 16, ...40.

Ans. As common difference is same so it is an AP and also last term is given. But we do not know the number of terms. Now as it is just a difference of 4 and the first term is 4 only, so here we can simply see, how many 4s make 40 i.e. 10, so the AP has 10 terms.

So we will apply the formula $S_n = x^2 + \frac{1}{x^2}$

Here a = 4, d = 4 and n = 10

$S_5 = \frac{10}{2}[4+40] \Rightarrow 5 \times 44 = 220$

Further, if three terms a, b, c are in AP, then $b = \frac{a+c}{2}$ zero.

2. Geometric Progression

A geometric progression is a sequence in which each term is derived by multiplying or dividing the preceding term by a fixed number called the common ratio.
e.g. the sequence 2, 4, 8, 16... is a Geometric Progression (GP) for which 2 is the common ratio

Now we will discuss important formulae of GP

(2a) General term of GP

The general form of a GP is a, ar, ar^2, ar^3 and so on
Thus nth term of a GP series is
$T_n = a r^{n-1}$
where a = first term and r = common ratio = T_m/T_{m-1}.

1. Find the 8^{th} term of series 2, 4, 8, 16....

Ans. As common ratio is same so it is a GP.
Here a = 2, r = 2 and n = 8

So applying the formula

$T_n = ar^{n-1}$

We get $T_8 = 2 \times 2^{8-1}$

$= 2 \times 2^7$

$= 2 \times 128$

$= 256$

(2b) Sum of n terms of an GP

There are 2 formulae of sum of n terms of a GP.

$$S_n = \frac{a(r^n - 1)}{r-1} \text{ when } r > 1 \ \& \ a\frac{(1-r^n)}{1} \text{ when } r < 1$$

2. Find the sum of first 8 terms of series 2, 4, 8, 16 ...

Ans. As common ratio is same, so it is a GP.

So we will apply the formula

$$S_n = \frac{a(r^n - 1)}{r-1}$$

Here $a = 2$, $r = 2$ and $n = 8$

$$S_8 = \frac{2(2^8 - 1)}{2-1} = \frac{2(256-1)}{2-1} = 2 \times 255 = 510$$

Further, if three terms a, b, c are in GP, then b = √ac

5 Percentage

1. Percentage

Percent is a term derived from Latin word 'per centum' which means 'per hundred' i.e. Percentages are numerators of fraction with denominator 100 and have been used in comparing results. Percentage is represented by the symbol % which means 'hundredths'. That is 1% means 1 out of a hundred or we can say one hundredth. It can be written as: $1\% = \frac{1}{100} = 0.01$

(1a) Conversion of Fraction to Percentage:

Multiply the fraction by 100 to convert it into percent.

1. Write $\frac{1}{6}$ as per cent:

Ans. We have, $\frac{1}{6} \times 100 = 16.67\%$

(1b) Conversion of Percentage to Fraction:

Reversing the earlier operation will convert a percent to a fraction i.e. divide the percent by 100.

2. Write 60% as fraction.

Ans. We have, $60\% = \frac{60}{100} = \frac{3}{5}$

(1c) Conversion of Percentage to "How many":

Let us understand this concept by taking an example.

3. A survey of 400 people showed that 25% like to read "The Hindus" newspaper. How many people like to read "The Hindus" newspaper?

Ans. Total no. of people = 400. Out of these, 25% like to read "The Hindus" newspaper.
Now, we can do this by two methods.
Method I: Out of 100, 25 like to read "The Hindus" newspaper. So, out of 400, number of people who like to read "The Hindus" newspaper

$$= \frac{25}{100} \times 400 = 100$$

Method II: Number of people like to read "The Hindu" newspaper = 25% of 400

$$= \frac{25}{100} \times 400 = 100$$

2. Increase or Decrease Percent

When we need to know the change in a quantity (increase or decrease) in the percentage form then we need to consider the old value and the new value and observe the percentage change. For example, if the score of a student increased from 550 to 605, then the increase in score is 55 which is 10% of 550, so the score increased by 10%.

It is worth remembering here, that the increase or the decrease will always be on the original quantity. If the increase or decrease is given in absolute and the percentage increase or decrease is to be calculated, then the following formula applicable will be

$$\%\ \text{increase/decrease} = \frac{\text{Quantity increase or decrease}}{\text{Original quantity}} \times 100$$

The point worth remembering is that the denominator is **the original quantity**.

4. The number of Spanish speaking people in a city K is 90000. Due to floods, some people migrated to other cities and number of Spanish speaking people left in the city is 65000. Find the percentage decrease in the number of Spanish speaking people.

1. $18\frac{7}{9}\%$ 2. $27\frac{7}{9}\%$ 3. $23\frac{7}{9}\%$ 4. $28\frac{7}{9}\%$

Ans. Initial number of Spanish speaking people = 90000
After floods, the number of Spanish speaking people = 65000
Decrease in the number of Spanish speaking people = 90000 – 65000 = 25000

Percentage decrease is given by $= \frac{\text{final} - \text{initial}}{\text{initial}} \times 100$

Percentage decrease in number of Spanish speaking people $= \frac{25000}{90000} \times 100 = \frac{250}{9} = 27\frac{7}{9}\%$

(2a) To Increase a Number by x %:

Let a number is increased by 10 %, then it will become 1.1 times of itself.
Similarly, if a number is increased by 20 %, then it will become 1.2 times of itself.
On the same lines, if a number is increased by 30 %, then it will become 1.3 times of what it was.

Illustrative Ex.: Let us take the number 60, if we have to increase it by 10% then we have
60 + 10% of 60 = 66 or 1.1 × 60 = 66

(2b) To Decrease a Number by x %:

Let a number is decreased by 10 %, then it will become 0.90 times of itself.

Similarly, if a number is decreased by 20 %, then it will become 0.80 times of itself.

On the same lines, if a number is decreased by 30 %, then it will become 0.70 times of what it was.

Illustrative Ex.: Let us take the number 60, if we have to decrease it by 10% then we have 60 – 10% of 60 = 60 – 6 = 54 Or 0.9 × 60 = 54

3. Salary/Weight/Income Comparison

- If A earns R % more than B, then B's earning is less than that of A by $100 \times \frac{R}{100+R}$ %
- If A earns R% less than B, then B's earning is more than that of A by $100 \times \frac{R}{100-R}$ %.

Note: If the question is about the price of a commodity that has got increased by a given percentage say, R %, then by what percent its consumption needs to be changed, so that the total expenditure remains fix. Then the way to solve such question is the same i.e. if the price is increased, then consumption should be decreased by $\frac{100 \times R}{(100+R)}$.

If the price is decreased, then consumption should be increased by $\frac{100 \times R}{(100-R)}$

4. Increase and Decrease by the same % age:

If a number happens to be first increased by *R* %, then this number is later decreased by *R* %, then in the net effect, total there would be a final decrease of $\frac{R^2}{100}$ %.

5. Increase and Decrease by different% age:

If a number happens to be first increased by a given percentage say, *X* %, then it is again increased by another given percentage say, *Y* %. Then the total net increase in this number will be $[X + Y + \frac{XY}{100}]$ %.

This formula is very important. It can be applied in so many other questions. In case instead of increase, there is a decrease, simply put a negative value in place of variable. You will get the right answer, even when both the decreases are mentioned. What you obtain after solving the formula, in case it is positive, there is an increase, and if it is negative, there is a decrease.

6. Equivalent Percentages of some commonly used Fractions:

Fraction	%age	Fraction	%age
1	100%	1/9	11.11%
1/2	50%	1/10	10%
1/3	33.33%	1/11	9.09%
1/4	25%	1/12	8.33%
1/5	20%	1/14	7.14%
1/6	16.67%	1/15	6.67%
1/7	14.28%	1/16	6.25%
1/8	12.5%	1/20	5%

6 Profit & Loss

Profit & Loss

Let us understand some definitions before going to the examples of the Profit & Loss.

1. Some Important Points

- **Cost Price (C.P):** The price at which article is purchased is called its Cost Price. All overhead expenses such as transportation, taxes etc. are also included in cost price.
- **Selling Price (S.P.):** The price at which article is sold is called its Selling Price.
- **Profit (P):** The amount gained by selling an article with more than its cost price (S.P > C.P) is called as profit, which is denoted by P.
  Profit = Selling Price – Cost Price

1. The cost price of article is Rs.300 and the selling price of article is Rs.360. Find the amount of profit.

Ans. Given, Cost Price = Rs.300
Selling Price = Rs.360
Since, S.P > C.P
Thus, Profit = S.P – C.P = Rs. 360 – 300 = Rs. 60

- **Loss (L):** You incur a loss, when you sell a product at a price which is lesser than the Cost price
  Loss = Cost Price – Selling Price

2. Profit and Loss Percent

Suppose that a dealer buys an article for Rs.100 and sells it for Rs.101, his profit is Rs.1. Another dealer buys an article for Rs.2 and sells it for Rs.3. His profit is also Rs.1. What did you think? Which transaction is better?

To compare the two rates of profit we represent them as a percentage. Thus, the percentage of profit in the first case is $\frac{1}{100}\times100=1\%$, and in second case is $\frac{1}{2}\times100=50\%$. In other words, the rates of profit are 1% and 50%. The latter is much higher than the former.

% Profit: $\%\ \text{Profit}=\frac{\textit{Total Profit}}{\textit{Cost Price}}\times100$

% Loss: $\%\ \text{Loss}=\frac{\textit{Total Loss}}{\textit{Cost Price}}\times100$

Gain or loss percent is never calculated on the number of articles sold. For example, if a dealer sells 100 articles and makes a profit of Rs.10, it is impossible to calculate his gain per cent unless the cost price of the articles can be found.

If 100 articles cost Rs.50, he gains $\frac{10}{50}\times100$, i.e. 20%

As, while calculating any percentage change, the increase or decrease is expressed as a percentage of first value, buying comes before selling, thus gain or loss is expressed as a percentage of the buying price (i.e., the cost price), not of the selling price.

If the gain is given to be 20%, it means that if the cost price is 100, the selling price is 120.

If the loss is given to be 20%, it means that if the cost price is 100, the selling price is 80.

So, in both the cases, it is the cost price that is 100, never the selling price.

2. If Sanjay bought a T.V. for Rs.300 and sold it for Rs.360, find the profit%.

Ans. Cost Price = Rs.300
Selling Price = Rs.360
Profit = Rs.60

$$\text{Profit }\% = \frac{60}{300}\times100=20\%$$

Note: **Profit and loss percentage is always calculated on cost price, unless otherwise specified.**

Now, we can understand that two out of the three quantities related to prices, i.e. C.P, S.P, amount of Profit or Loss or their percentage, we can find the rest.

3. The cost of a toy is Rs.120. If the shopkeeper sells it a loss of 10%, find the price at which it is sold.

Ans. Cost Price = Rs.120
Loss of 10 % means if C.P is Rs.100, Loss is Rs.10
Therefore, S.P would be Rs. (100 – 10) = Rs.90
When C.P is Rs.100, S.P is Rs. 90. Therefore, if C.P were Rs.120 then S.P = $\frac{90}{100}\times120=Rs.108$

Otherwise, Loss is 10% of the cost price
= 10% of 120 = Rs.12
⇒ S.P. = C.P. – Loss = Rs.120 – Rs.12 = Rs.108.

3. Marked Price

The price which is on the label of an article or product is called its 'Marked Price' or 'List Price'. This is the price at which the shopkeeper intends to sell the product.

Discount: It is a reduction amount made from the list price or marked price.

Thus, Marked Price – Discount = Selling Price.

So, we can say that Discount = Marked Price – Selling Price = Discount % = $\frac{\text{Discount}}{\text{Marked Price}} \times 100$

4. An item marked at Rs.2000 is sold for Rs.1800. What is the discount and discount %?

Ans. Discount = Marked Price – Selling Price

= Rs.2000 – Rs.1800 = Rs.200

Since discount is on Marked Price, we will have to use marked price as the base.

On Marked Price of Rs.2000, the discount is Rs.200

On M.P of Rs.100, the discount will be

$= \frac{200}{2000} \times 100 = 10\%$

4. Successive Discounts

Successive discount is the discount which is offered on the discount.

Hurry up!!!!
40% + 10%
Off!!!

To understand the concept of successive discount in detail, consider a situation:

A lot of times while crossing a grocery shop or a garment store, you must have seen a poster saying 'Hurry, 40% + 10% off!'

Now, you might think that the total discount is '40 + 10 = 50%', which is wrong as this doesn't refers to total discount 50% discount, whereas it refers that there will be discount of 40% on marked price and further 10% discount will be given on the discounted price. Suppose you purchased a product whose marked price is Rs.1000 and offer on the product is 40% + 10% means that 40% discount will be available on Rs.1000 i.e. discount of Rs.400, further 10% discount will be available on Rs.600 i.e. discounted amount which will be Rs.60, after that you purchase the product for Rs.540.

So, the total discount = 1000 – 540 = 460

Discount % = $\frac{460}{1000} \times 100 = 46\%$

Or a straight method can be applied for two discounts. Single discount, which is equal to two successive discounts of x % and y% = $\left(x + y - \frac{xy}{100}\right)\%$

So, we can also apply this formula to get resultant i.e.

$$= \left(40 + 10 - \frac{40 \times 10}{100}\right)\% = 46\%$$

5. The marked price of a jacket is Rs.1000. The successive discounts offered by the store were 20% and 30%. Find the single discount equivalent to the two discounts given by the shopkeeper.

Ans. Successive discounts offered = 20% & 30%.

S.P = 1000 – 20% of 1000 = 800

800 – 30% of 800 = 560.

Total discount = 1000 – 560 = 440

Discount % = $\frac{440}{1000} \times 100 = 44\%$

Otherwise, Total discount

$$= 20 + 30 - \frac{20 \times 30}{100} = 44\%$$

Now, it is clear that in case of successive discount you can avail any of the discount on marked price and other on discounted price the resulting amount will be the same i.e. either you avail 30% discount on marked price and then 20% discount on the discounted price or you can avail 20% discount on the marked price and then 30% discount on the discounted price resulting in the same total discount. Same will be applicable in case of 3 successive discounts.

Dishonest Dealers and Faulty weights:

When a tradesman professes to sell at cost price, but uses a false weight, then the percentage profit earned

$$= \frac{\text{Error}}{\text{True Weight- Error}} \times 100$$

7 Simple & Compound Intrest

Simple Interest

1. Simple Interest

If the interest is calculated on the original money borrowed throughout the period for which loan was taken, which means there is no change in the principal, and then the interest is called simple interest.

The formula to calculate the Simple Interest is ;

$$S.I = \frac{P \times R \times T}{100}$$

Principal (P): The money borrowed or lent.

Interest (I): It is the additional money paid to the lender, for the use of the money borrowed.

Rate (R): Rate of Interest in % per annum

Time (T): The time period for which the money is borrowed.

Amount (A): Principal + Interest

1. Find the amount to be paid by Alexa to Batty after the end of 2 years if Alexa has taken a loan of Rs. 5000 at the rate of 20% per annum, calculated half-yearly at simple interest from Batty.

1. Rs. 1500 2. Rs. 2000
3. Rs. 2500 4. Rs. 3000

Ans. Here P = Rs. 5000, T = 2 years = 4 half-years,

$$R = \frac{20\%}{2} = 10\% \text{ half-yearly}$$

$$\text{Simple interest} = \frac{P \times R \times T}{100} = \frac{5000 \times 10 \times 4}{100}$$

= Rs 2000. Option (b)

(1a) Formula for Simple Interest

To find simple interest on Rs. P (Principal) for T years at rate of interest which is R% per annum.
The interest on Rs. P for 1 years is R% of P, i.e.,

$$Rs.\ A = P\left(1+\frac{R}{200}\right)^{2n} = Rs.\frac{R \times P}{100}$$

∴ The interest on Rs. P for T years is

$$\text{Rs.}\left(\frac{R \times P}{100}\right) \times T = \text{Rs.}\frac{P \times R \times T}{100}$$

Hence, we have the following formula:

$$S.I = \frac{P \times R \times T}{100}$$

2. In how many years will Rs. 900 yield an interest of Rs. 324 at 12% per annum simple interest?

Ans. Given, P = Rs. 900, R = 12% p.a.
S.I. = Rs. 324

Using the formula, $S.I = \frac{P \times R \times T}{100}$, we get,

$$324 = \frac{900 \times 12 \times T}{100} = 108T$$

$\Rightarrow$ $T = 3$ years

Important Notes:

(i) When time is given in days, then we need to convert it into years by dividing it by 365.

(ii) When time is given in months, we convert it to years by dividing it by 12.

(iii) When dates are given, the day on which the sum is borrowed is not included but the day on which the money is returned back is always included, while counting the number of days.

(iv) If Rate is 1.5% per month that means Rs.1.50 for every Rs.100 per month = Rs.1.5 × 12 = Rs.18 for every Rs.100 per year = 18% p.a.

Compound Interest

1. Compound Interest

As we know how to calculate the simple interest, but in normal business circles, simple interest is rarely used. Suppose a man deposits Rs.1000 in a saving account at an interest rate of 10% per annum. At the end of one year he will get Rs.100 interest on his deposit. However, unless he takes out his Rs.100 in cash, it will be added to his original Rs.1000. Thus, if he leaves his money in his account, in the next year the bank will be paying him interest on his original Rs.1000 plus the Rs.100 interest, i.e., on Rs.1100. In the third year the interest will once again be added to the new principal on Rs.1100, and so on for a long as the money is left in the account. This kind of interest is known as **compound interest.**

1. Find the compound interest on Rs. 1000 for 3 years at 10 % per annum.

Ans. Step I: Principal for the first year = Rs.1000

Interest for the first year

$$= Rs.\frac{1000 \times 10 \times 1}{100} = Rs.100$$

$\therefore$ Amount at the end of first year

= Rs. 1000 + Rs.100 = Rs.1100

Step II: Principal for the second year = Rs.1100

Interest for the second year

$$= Rs.\frac{1100 \times 10 \times 1}{100} = Rs.110$$

$\therefore$ Amount at the end of second year

= Rs.1100 + Rs.110 = Rs.1210

Step III: Principal for the third year = Rs.1210

Interest for the third year

$$= Rs.\frac{1210 \times 10 \times 1}{100} = Rs.121$$

$\therefore$ Amount at the end of third year

= Rs.1210 + Rs.121

= Rs.1331

$\therefore$ Compound Interest at the end of 3 years

= 1331 – 1000 = Rs.331

(1a) Formula for Compound Interest:

Important Results: $A = P\left(1+\frac{R}{100}\right)^n$ and

$$C.I. = A - P = P\left(1+\frac{R}{100}\right)^n - P = P\left[\left(1+\frac{R}{100}\right)^n - 1\right]$$

2. Find the compound interest on Rs. 5000 for 3 years at 10% per annum compounded annually.

Ans. Given,

Principal = Rs. 5000,

Time = 3 years,

Rate = 10%

As, we know $A = P\left(1+\frac{R}{100}\right)^n$

$$= 5000\left(1+\frac{10}{100}\right)^3$$

$$= 5000\left(\frac{11}{10}\right)^3 = 5000 \times \frac{11}{10} \times \frac{11}{10} \times \frac{11}{10} = 6655$$

C.I = A – P = 6655 – 5000 = 1655

Now, when the rate of interest for successive years are different.

If the rate of interest is different for every year say R_1, R_2, R_3 for the first, second, third year, then the amount after 3 years is given by

$$A = P\left(1+\frac{R_1}{100}\right)\left(1+\frac{R_2}{100}\right)\left(1+\frac{R_3}{100}\right)$$

3. Find the compound interest on Rs.80000 for 3 years if the rates are 4%, 5% and 10% respectively.

Ans. $A = 80000\left(1+\frac{4}{100}\right)\left(1+\frac{5}{100}\right)\left(1+\frac{10}{100}\right)$

$$= 80000 \times \frac{104}{100} \times \frac{105}{100} \times \frac{110}{100} = Rs.96096$$

$\therefore$ Compound Interest = 96096 – 80000 = Rs.16096

Now, there are some questions in which we have to calculate the difference between compound interest and simple interest.

If the difference between the two kinds of interests, that is between compound interest and simple interest is of two years then, **Difference** = $P\left(\frac{R}{100}\right)^2$

If the difference between the two kinds of interests, that is between compound interest and simple interest is of three years then,

Difference = $3 \times P\left(\frac{R}{100}\right)^2 + P\left(\frac{R}{100}\right)^3$

4. Find the difference between compound interest and simple interest on Rs. 1000 for 2 years at 10% p.a.

Ans. Principal = Rs. 1000,

Time = 2 years,

Rate = 10%

$$S.I = \frac{P \times R \times T}{100} = \frac{1000 \times 10 \times 2}{100} = 200$$

$$C.I = A - P = A = P\left(1+\frac{R}{100}\right)^n - P$$

$$= 1000\left(1+\frac{10}{100}\right)^2 - 1000$$

= 1210 – 1000

= 210.

Difference = C.I – S.I = 210 – 200 = 10

Or by formula,

$$\text{Difference} = P\left(\frac{R}{100}\right)^2 = 1000 \times \left(\frac{10}{100}\right)^2 = 10$$

5. If Rs.2000 becomes Rs.2400 in 7 years at a certain rate of compound interest, how much it will become in 14 years?

Ans. Compound interest in 7 years

$$= 2400 - 2000$$
$$= \text{Rs.}400$$

Rate of interest in 7 years

$$= \frac{400}{2000} \times 100 = 20\%$$

In 14 years, there will be two periods of 7 years, so $n = 2$

$$A = P\left(1 + \frac{R}{100}\right)^n = 2000\left(1 + \frac{20}{100}\right)^2 = \text{Rs.}2880$$

6. Raman bought a TV for Rs.16000. If the cost of the television is depreciating at the rate of 5% per annum, calculate its value after 2 years.

Ans. Present value of Television (P) = Rs.16000 Rate of depreciation (R) = −5% (Negative sign indicates for depreciation) Time = 2 years. Therefore, the value of television after 2 years $= 16000\left(1 - \frac{5}{100}\right)^2$

$= 16000 \times \frac{19}{20} \times \frac{19}{20} = 14440$ a false weight, then the

$$\text{percentage profit earned} = \frac{\text{Error}}{\text{True Weight- Error}} \times 100$$

8 Chain Rule, Time & Work

1. Chain Rule

The concept of Chain Rule is very important as it has application in almost all the chapters of mathematics. In order to understand the concept of chain rule or unitary method, let us take some examples:

1. Mohan bought 2 kg apples for Rs 150. What will be the price of 5 kg apples?

Ans. When you come across such kind of problems then the chain rule or unitary method is used. This method simply means that we are making a chain. First of all, try to find the cost of one article and then multiply it with required number.

In the above example, the cost of 2 kg apples = Rs 150

The cost of one kg apples = $\frac{150}{2}$ = Rs 75

Therefore, the cost of 5 kg apples = 75 × 5 = Rs 375

As here the value of one unit is found and then the value of the required number of units is found, so it is called unitary method.

In the above example, the unitary method or the chain rule used is the direct method. It means that when the number of units increases, the value increases and when the number of units decrease, the value also decreases.

But when it comes to the problems like some men are doing a work, then the above unitary method changes. To understand this better, let us assume that a work is to be done by 10 men and they take 4 days to complete it. So, does it mean that 20 men will take 8 days to complete it? Obviously not. So, what is the catch here? Actually, when the number of men increases, the number of days will decrease as more men will do the same work in lesser number of days. Let us check how that will vary?

Initially the work is done by 10 men in 4 days. Now if the number of men becomes double, then the number of days will become half. So, when 20 men are employed on the work, it will take 2 days to complete the work. If the number of men doing the work is again doubled, then the time taken will be halved again. So, if 40 men are employed on the task, so time taken will be just one day.

Now based on the above discussion we get one important result here. If you look closely then you will find that the product of the number of men and the number of days is constant. In the first case it is 10 × 4 = 40, in the second case it is 20 × 2 = 40 and in the third case it is 40 × 1 = 40. This constant product of the men and days is normally taken as the total work.

Let us understand it with another example.

2. If 10 men can do a work in 8 days, in how many days the same work will be completed by 16 men?

Ans. As we discussed above, the product of the number of men and the total days will remain constant and can be considered as the total work, so here we can assume that the total work is 10 × 8 = 80 units. In the second case, the number of men are 16 and let they take 'x' days to complete the work, so the product of these men and the number of days will be equal to the total work.

Therefore, we have $16 \times x = 80 \Rightarrow x = 5$ days

Hence, the same work will be completed by the 16 men in 5 days.

3. If 10 men can do a work in 12 days working 7 hours daily, then in how many days, 14 men working 10 hours daily, will complete the same work?

Ans. In the first group, there are 10 men who can do the work in 12 days working 7 hours daily.

So, the total work can be considered as 10 × 12 × 7 = 840 units.

Let the second group takes 'x' days to complete the work.

The total work for the second group is $14 \times x \times 10 = 140x$ units.

As the work is same in both the cases, therefore, $140x = 840 \Rightarrow x = 6$.

Therefore, the second group takes 6 days to complete the work.

In the above mentioned discussion, we have taken the work done by different groups to be same. Sometimes questions can be asked where the work done is different. In that case we have a simple relation which is as follows:

If M_1 men can do W_1 work in D_1 days working H_1 hours daily and M_2 men can do W_2 work in D_2 days working H_2 hours daily then

$M_1 \times D_1 \times H_1 \times W_2 = M_2 \times D_2 \times H_2 \times W_1$

Here the work done by the two groups is written on the other side of the equation. The work done (W_1) by the first group is written on the side of the M_2, D_2 and H_2 and same is done for W_2.

The above relation also holds if the work done by the two groups is equal. In that case $W_1 = W_2$ and we get the same relation which we discussed in the above examples.

Note: In the above relation, if one or other parameter is not given (e.g., if the number of hours are not given) then that parameter can be dropped from it.

2. Time & Work

If $\frac{3}{5}^{th}$ of a work is done in 18 days, then in how many days $\frac{5}{6}^{th}$ of that work will be done?

Here, $\frac{3}{5}^{th}$ of the work is done in 18 days. So the complete work will be done in $\frac{5}{3} \times 18 = 30$ days.

Therefore, $5/6^{th}$ work will be done in $30 \times \frac{5}{6} = 25$ days.

How will this work when there are more people working together or at different stages of the work. Let us understand all that with the help of some examples.

1. If A and B can do the work in 10 days and 15 days respectively then in how many days the work will be completed if they work together?

Ans. Whenever, there are two or more than two persons working on the same work together, find the work done by them per day and add that work to get the total work done by them per day.

Here, A and B can do the work in 10 days and 15 days respectively. As discussed above, work done by A in one day is $\frac{1}{10}^{th}$ of the total work and work done by B in one day is $\frac{1}{15}^{th}$ of the total work.

Therefore, when they work together, the work done by them in one day

$$= \frac{1}{10} + \frac{1}{15} = \frac{5}{30} = \frac{1}{6}.$$

Therefore, together they can do $1/6^{th}$ of the total work. Hence, the work will be completed in 6 days.

There is another approach we can apply and you will find it easy to apply.

In this example, we have used the fractions as the total work was considered to be 1 unit. But we can get rid of these fractions if we use the total work equal to some common multiple (say LCM) of the number of days taken by the persons individually.

The times taken by A and B individually are 10 and 15 days respectively. Now consider the total work as the LCM of 10 and 15 i.e. 30 units (although you can take any common multiple of these two numbers).

If the total work is 30 units and A is doing it in 10 days, then it means that A is doing $\frac{30}{10} = 3$ units per day.

Similarly, B is doing $\frac{30}{15} = 2$ units per day.

When they work together, then they will do $3 + 2 = 5$ units per day. As the total work is 30 units, therefore, together they will complete it in $\frac{30}{5} = 6$ days.

Let us now consider the cases where few people start the work and after some time some of them leave the work.

2. Three people A, B and C can do the work in 15, 20 and 30 days respectively. They start the work together but after 2 days C left the work. In how many days the work will be completed?

Ans. Let us consider the total work as the LCM of 15, 20 and 30 i.e. 60 units.

Work done by A per day = $\frac{60}{15} = 4$ units

Work done by B per day = $\frac{60}{20} = 3$ units

Work done by C per day = $\frac{60}{30} = 2$ units

Work done by A, B and C together per day = 9 units

Now in the first 2 days all of them worked together, so the work done = $9 \times 2 = 18$ units

After that C left and the remaining work will be completed by A and B together.

The remaining work = $60 - 18 = 42$ units.

A and B can do $4 + 3 = 7$ units/day

So they will do the remaining work in $\frac{42}{7} = 6$ days.

Therefore, total work is done in $6 + 2 = 8$ days.

3. Pipes And Cisterns

The problems on pipes and cistern are almost same as we discussed above for Time and Work.

The only difference is, here instead of men, pipes are doing the work to fill or empty the tanks.

1. A pipe can fill the tank in 15 hours but due to the leakage at the bottom it takes 5 hours more to fill it. In how much time the leakage can empty the filled tank?

Ans. Let the leak can empty the tank in X hours. Water filled by pipe in one hour = $\frac{1}{15}$

Water leaked in one hour = $\frac{1}{X}$

As it took 5 hours more to fill the tank, so the tank was filled in 20 hours.

Now we have the following equation,

$$\frac{1}{15}-\frac{1}{X}=\frac{1}{20} \Rightarrow \frac{1}{X}=\frac{1}{15}-\frac{1}{20}=\frac{1}{60}$$

Therefore, X = 60 hours.

Alternate method: Let us try this question by the second method. The time taken by the pipe is 15 hours and actually it took 20 hours to fill the tank. Let us consider the total capacity of the tank as LCM (15, 20) = 60 litres.

Now the rate of filling of the pipe

$$= \frac{60}{15}$$

= 4 litre/hour.

The actual rate of filling is $\frac{60}{20}$ = 3 litre/hour

Now what does it mean? Pipe is pouring in 4 litre in one hour but net inflow is 3 litre/hour. So where does 1 litre/hour go?? Obviously, that is drained by the leak. So the leak is draining at the rate of 1 litre/hour. Hence, to drain the complete tank of 60 litres, it will take 60 hours.

2. Two pipes A and B can fill a cistern in 20 and 24 minutes respectively. Both pipes being opened, find when the first pipe must be turned off, so that the cistern may be filled in 12 minutes?

Ans. Since the cistern is to be filled in 12 minutes, Second pipe can fill only $\frac{12}{24}=\frac{1}{2}$ of the cistern in total time. This means the other half must be filled by the first pipe. The first pipe can fill the whole tank in 20 minutes, so half of the tank it can fill in half of the 20 minutes i.e. 10 minutes. Now the first pipe is opened from the beginning, it should be turned off after 10 minutes.

| 9 | Ratio & Problems on Ages

1. Ratio & Partnership

Consider the following:

Lalita's weight is 20 kg and her father's weight is 80 kg. How many times the father's weight is of Lalita's weight? Yes, it is four times.

Cost of a copy is Rs 3 and cost of a book is Rs 15. How many times the cost of a book as compared to the cost of a copy? Yes, it is five times.

In the above examples, we compared the two quantities in terms of 'how many times'. This comparison is called Ratio. We denote ratio using symbol ':'

Now consider the following example.

Length of a snake is 50 cm and the length of a crocodile is 5 m. Does it mean that the snake is 10 times bigger than the crocodile? Absolutely not. So what is wrong here? Observe that the length of the snake is in centimeters and length of the crocodile is in meters.

So, we have to convert their lengths to the same units.

Length of the crocodile = 5 m = 5 × 100 = 500 cm.

Therefore, the ratio of the length of the crocodile to the snake = $\frac{500}{50} = \frac{10}{1} = 10 : 1$.

Hence, two quantities can be compared only if they are in the same unit.

Solved Examples:

1. There are 120 teachers in a school of 4200 students. Calculate the ratio of the number of students to the number of teachers.

Ans. The number of teachers = 120.
The number of students = 4200.

Required ratio = $\frac{4200}{120} = 35:1$

2. Ram and Sham divided Rs 800 between them in the ratio 3: 5. How much money will each get?

Ans. Here the ratio in which the total money is divided is 3: 5. So let the money that Ram got is 3x and that obtained by Sham is Rs 5x.

As the total is Rs 800, so we have, $3x + 5x = 800$

$\Rightarrow \quad 8x = 800$

$\Rightarrow \quad x = 100$

Hence the money got by Ram = 3x = 3 × 100
= Rs 300

The money got by Sham = 5x
= 5 × 100 = Rs 500.

Alternative Method: Here the total money divided is Rs 800 and the ratio in which it is divided is 3 : 5. Now this ratio 3: 5 simply means that if Ram gets Rs 3 then Sham will get Rs 5 and the total money is Rs 8.

Now we can use the unitary method.
If the total money is Rs 8, then Ram will get Rs 3.
If the total money is Rs 800, then Ram will get

$$\text{Rs.}\frac{3}{8} \times 800 = \text{Rs.}300$$

Similarly Sham will get Rs. $\frac{5}{8} \times 800 = Rs.500$

Now here, we can also interpret that in the numerator, the part of the ratio is taken and in the denominator, the sum of the ratio is taken and the fraction so obtained is multiplied by the total value.

3. Rs 1500 are divided between A, B and C in the ratio 2: 3: 5. By how much percent is the share of C more than the share of A?

Ans. As discussed above, the share of A

$$= \frac{2}{10} \times 1500 = \text{Rs.}300\,.$$

The share of C = $\frac{5}{10} \times 1500 = \text{Rs.}750$

The required percentage

$$= \frac{750 - 300}{300} \times 10 = \frac{450}{300} \times 100$$
$$= 150\%$$

Alternative method: Now as in this question, only percentage value is asked, so there is no need to find the shares of A and C separately.
We can see that in the ratio, the share of A is 2 parts and that of C is 5 parts.
So we can directly say that the share of C will be more than the share of A by $\frac{5-2}{2} \times 100 = 150\%$

2. Variation & Proportion

(2a) Variation

(i) Direct Variation (Proportion):
If in two variables the relation is such that increase in one result in increase in other and decrease in one result in decrease in other they are said to be in Direct Variation or Direct Proportion
Suppose that the price of tea is Rs. 30 per kg. Then the cost of 2 kg of coffee will be Rs. 60. i.e. as the quantity of coffee increases, the total cost of the coffee purchased also increases.

(ii) Inverse Variation (Proportion):
If in two variables the relation is such that increase in one result in decrease in other and decrease in one result in increase in other they are said to be in Inverse Variation or Inverse Proportion
Suppose 6 men can do a piece of work in 12 days, then one man will do it in $6 \times 12 = 72$ days.

Let us now learn some properties of ratio:

- Ratio itself have no units but is the comparison of two quantities having same units
- When numerator and denominator of a ratio are multiplied by same number, the ratio remains the same
- When two different ratios, $a : b$ and $c : d$ are expressed in different units, then to obtain the combined ratio, we multiply the two.
- Compounding of $a : b$ and $c : d$ gives the result as $\frac{a \times c}{b \times d}$.

(2b) Proportion

If $a : b = c : d$, we read this equality as "*a is to b as c is to d*" and say that *a, b, c, d* are in proportion.
It means when two ratios are same, they are in proportion
If $a : b = c : d$, then $ad = bc$.

Continued Proportion:
If middle term is same i.e. $a : b = b : c = c : d$, then the four numbers are said to be in continued proportion.

Solved Examples:

1. If m is proportional to n and m = 2, when n = 3, find the value of m when n = 15.

Ans. When m = 2, n = 3. Also m = kn

$$\Rightarrow \quad 2 = 3k \ \Rightarrow k = \frac{2}{3}.$$

When n = 15, $m = \frac{2}{3} \times 15 = 10$

2. A sum of money is to be distributed among A, B, C, D in the proportion of 5:2:4:3. If C gets Rs. 1000 more than D, what is B's share?

Ans. Let the shares of A, B, C and D be Rs. 5x, Rs. 2x, Rs. 4x and Rs. 3x respectively.
Then, $4x - 3x = 1000$
$\Rightarrow \quad x = 1000.$
Hence B's share = Rs. 2x = Rs. 2×1000 = Rs. 2000

3. Partnership

Ratio of profits = Ratio of investment

Let us learn the concept with the help of Example:

Solved Examples:

1. A and B are partners in a business. A invests Rs 300 for 12 months and B invested Rs 800 for 6 months. If they gain a profit of Rs 1400 at the end of one year, what is A's share?

Ans. A's total capital = 300×12 = Rs 3600.
B's total capital = 800×6 = Rs 4800.
Profit sharing ratio = 3600 : 4800
$\Rightarrow$ 3 : 4.
The profit is given to be Rs. 1400
The share of A = $1400 \times \frac{3}{7}$ = Rs 600

2. X and Y invest in a business in the ratio 3: 2. They decided that 5% of the total profit will go to charity and X's share is Rs. 855. Find the total profit.

Ans. Let the total profit is Rs 100. After paying to the charity, remaining profit is Rs 95.
Share of X in the remaining profit = $\frac{3}{5} \times 95$ = Rs 57
Therefore, if A's share is Rs 57, total profit = Rs 100
If A's share is Rs 855, total profit
$= 100 \times \frac{855}{57}$ = Rs 1500

3. P, Q and R invested Rs. 50,000 for a business. P invested Rs. 4000 more than Q and Q invested Rs. 5000 more than R. Out of a total profit of Rs. 35,000, find the share of P.

Ans. Let the investment made by R is Rs x, by Q is Rs (x + 5000) and by P is Rs (x + 5000 + 4000) = Rs (x + 9000)
Total investment by P, Q and R = Rs 50000
$\Rightarrow$ x + x + 5000 + x + 9000 = 50000
$\Rightarrow$ 3x + 14000 = 50000

$\Rightarrow \qquad 3x = 36000$ or $x = 12000$

Therefore, the investment of P

$= x + 9000 = $ Rs 21000

Investment of Q $= x + 5000 =$ Rs 17000

Investment by R = Rs 12000

Ratio of profits = Ratio of investment. Therefore, the ratio of profits of P, Q and R = 21: 17: 12

Hence, that share of P in the total profit

$$= 21 \times \frac{35000}{50} = \text{Rs } 14700$$

4. A, B and C hired a ground for Rs 12000. A used this ground for 6 cows for 4 weeks, B used it for 12 cows for 4 weeks and C used it for 9 cows for 8 weeks. What amount of rent should C pay?

Ans. A's total use $= 6 \times 4 = 24$.

B's total use $= 12 \times 4 = 48$.

C's total use $= 9 \times 8 = 72$.

Ratio of their expenditure $= 24 : 48 : 72 \Rightarrow 1 : 2 : 3$.

$\Rightarrow$ C should pay $\frac{3}{6}$ of the rent i.e. $12000 \times \frac{3}{6}$

$=$ Rs 6000.

4. Problems on Ages

The trick to solve the problem on ages is to understand the language well and try to frame the equations by considering the required variables. Note, kindly take care of the tenses (is, was, ago, hence, etc.) while solving such problems.

Solved Examples:

1. The ages of Ray and Hari are in the ratio 5:7. Four years later the sum of their ages will be 56 years. What are their present ages?

Ans. Let the present age of Ray $= 5x$ and the present age of Hari $= 7x$

Four years later, age of Ray $= 5x + 4$ and age of Hari $= 7x + 4$

Given, $5x + 4 + 7x + 4 = 56$

$\Rightarrow 12x + 8 = 56$

$\Rightarrow 12x = 48 \Rightarrow x = 4$

Hence, present age of Ray $= 5x = 20$ years and present age of Hari $= 7x = 28$ years

2. Bikram father is 26 years younger than Bikram's grandfather and 29 years older than Bikram. The sum of the ages of all the three is 135 years. What is the age of each one of them?

Ans. Let the age of Bikram = x years

$\Rightarrow$ The age of Bikram's father $= x + 29$

$\Rightarrow$ The age of Bikram's grandfather

$= x + 29 + 26$

$= x + 55$

Also given, $x + x + 29 + x + 55 = 135$.

Solving, $3x = 51 \Rightarrow x = 17$

Hence, the age of Bikram = 17 years, age of Bikram's father = 46 years and age of Bikram's grandfather = 72 years

3. Fifteen years from now, Ravi's age will be four times his present age. What is Ravi's present age?

Ans. Let the present age of Ravi = x. Fifteen years from now, Ravi's age $= x + 15$. This is given to be four times his present age. Hence $x + 15 = 4x$.

Solving, $3x = 15 \Rightarrow x = 5$

Hence, the present age of Ravi = 5 years

4. Sheena's mother's present age is six times Sheena's present age. Sheena's age five years from now will be one third of his mother's present age. What are their present ages?

Ans. Let the present age of Sheena = x

$\Rightarrow$ The present age of Sheena's mother $= 6x$

Sheena's age five years from now $= x + 5$.

This is given to be one third of his mother's present age Hence $x + 5 = \left(\frac{1}{3} \times 6x\right) = 2x$. Solving, $x = 5$

Hence, Sheena's present age = 5 years and Sheena's mother's present age = 30 years

10 Time, Speed & Distance

Time Speed & Distance Basics

The topic starts with the basic formula i.e.

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

From this we can derive the following:

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

or $\quad$ Distance = Speed × Time

Units: The two units of speed that are normally taken **kilometer per hour (km/h) and meter per second (m/s)**

$$1 \text{ km/hr} = \left(\frac{1\times1000}{1\times3600}\right) = \frac{5}{18} \text{ m/sec}$$

For conversion of speed from **km/hr to m/sec, multiply** the speed by $\frac{5}{18}$.

For conversion of speed from **m/sec to km/h, multiply** the speed by $\frac{18}{5}$.

The topic is divided into four parts

I. Basics of Speed, Time and Distance
II. Relative Speed & Trains
III. Boats & Streams
IV. Races

Let's discuss these one by one

1. Speed, Time and Distance

In this we have to find the missing parameter out of three (speed, time and distance) when two others are given. The units should be compatible.

Solved Examples:

1. What is the distance covered by a car (in km) traveling at a speed of 84 km/hr in 10 minutes?

Ans. Distance covered = $84 \times \frac{10}{60} = 14$ km (Time is to taken in hours as speed is km/h)

2. A bus is moving at a speed of 100 km/hr, how long (in min) is it going to take to travel 25 km?

Ans. Distance = Speed × Time

$\Rightarrow$ Time = Distance ÷ Speed

$\Rightarrow \frac{25}{100} = 0.25$ hours = 15 min

Note: While converting decimal hours into the minutes, these are to be multiplied with 60 and not by hundred.

3. A man covers 75 km in 90 minutes. What is his speed in km/h?

Ans. Speed = Distance ÷ Time. Since, time is given in minutes and required answer is in km/h, so we need to convert time into equivalent hours. 90 minutes $= \frac{90}{60} = 1.5$ hours.

Therefore, speed $= \frac{75}{1.5} = 50$ km/h

4. Walking at $\frac{3}{5}th$ of his usual speed, Ravi reached his destination 20 minutes late. Find his usual time.

Ans. $\frac{3}{5}$ of usual speed implies $\frac{5}{3}$ of usual time (ut)

$\frac{5}{3}$ of ut = ut + 20 min $\Rightarrow \frac{5}{3}$ of ut − ut = 20 min

$\Rightarrow \frac{2}{3}$ of ut = 20 min

$\Rightarrow$ ut = 30 min

2. Average Speed

(i) When distance is same for two different speeds

Let the two speeds be a km/hr and b km/hr.
Let the distance traveled at each of these speeds be x km.

Time taken to cover x km at 'a' km/hr $= \frac{x}{a}$ and at 'b' km/hr $= \frac{x}{b}$.

Total time taken $= \frac{x}{a} + \frac{x}{b} = \frac{bx+ax}{ab} = \frac{x(b+a)}{ab}$, and the total distance covered = $2x$.

Therefore, average speed $= \dfrac{2x}{\frac{x(a+b)}{ab}} = \frac{2ab}{a+b}$.

(ii) When time is same for two different speeds

The average speed of traveling at two different speeds for the same time duration is just the simple average of these two speeds. Average speed = $\dfrac{(a+b)}{2}$.

5. While travelling from home to school, a student travelled at the speed of 75 km/hr. While coming back home from school on the same path, he travelled at the speed of 25 km/hr. What is my average speed of travel?

Ans. Average speed = $\dfrac{2ab}{a+b} = \dfrac{(2\times75\times25)}{100} = 37.5$ km/hr

6. A motorist travels for one hour at an average speed of 60 km/hr and next hour at an average speed of 70 km/hr. Then his average speed is

Ans. Average speed = $(60 + 70) \div 2 = 65$ km/hr.

3. Relative Speed and Trains

(3a) Relative Speed

Let the speed of two objects be x and y

(i) **In same direction,** their relative speed is the difference between the two speeds.
Relative speed = $x - y$ (considering $x > y$)

(ii) **In opposite direction,** their relative speed is the sum of the two speeds.
Relative speed = $x + y$

(3b) Some more facts:

(i) **If a train crosses a stationary man / lamp post / sign post /** then the distance travelled is the length of the train.

(ii) **If a train crosses a platform / bridge** then the distance traveled is the **length of the train + length of the object**

(iii) **If a train crosses a car / bicycle / a moving man -** in these cases, the relative speed between the train and the object is taken depending on the direction of movement of other object relative to the train and the distance traveled is the length of the train.

(iv) **If a train crosses another train which is moving at a particular speed in the same or opposite direction** – in these cases, the other train is moving as well and relative speed between them is taken depending on the direction of the other train and distance travelled is the sum of the lengths of both the trains.

Solved Examples:

1. A train traveling at 60 km/hr crosses a man in 6 sec. Calculate the length of the train in metres.

Ans. Speed in m/sec = $60 \times \dfrac{5}{18} = \dfrac{50}{3}$ m/sec.

Time taken to cross the man = 6 seconds.

Hence distance traveled = $\dfrac{50}{3} \times 6 = 100$ meters = length of the train.

2. A train traveling at 60 km/hr crosses another train traveling in the same direction at 50 km/hr in 45 sec. What is the total combined length of both the trains?

Ans. Speed of train A in m/sec = $60 \times \dfrac{5}{18} = \dfrac{50}{3}$ m/sec

Speed of train B in m/sec = $50 \times \dfrac{5}{18} = \dfrac{125}{9}$ m/sec

The relative speed is = $\dfrac{50}{3} - \dfrac{125}{9} = \dfrac{25}{9}$ m/sec.

Time taken for train A to cross train B = 45 seconds.

Therefore, distance traveled = $\dfrac{25}{9} \times 45 = 125$ meter = Combined length of two trains.

3. A train crosses a platform of length 300 m in 30 seconds and overtakes a car traveling in the same direction at speed of 90 km/hr in 120 seconds. What is the length of train (in m) and the speed at which it is traveling (in km/hr)?

Ans. Let the length of the train = x m and speed of the train = s km/hr
We know, distance = speed × time

Given, $(x + 300) = s \times \dfrac{5}{18} \times 30$

$\Rightarrow \quad x = s \times \dfrac{5}{18} \times 30 - 300$

Also, since the car is travelling in the same direction, $x = (s - 90) \times \dfrac{5}{18} \times 120$

Solving, $s \times \dfrac{5}{18} \times 30 - 300 = s \times \dfrac{5}{18} \times 120 - 3000$

$\Rightarrow \quad 25s = 2700 \Rightarrow s = 108$ km/hr and $x = 600$ m
So, length of the train = 600 m and speed of the train = 108 km/hr

4. A train traveling at 108 km/hr crosses a bridge of 600 m length completely in 30 seconds. What is the length of the train?

Ans. Speed = 108 km/hr = $108 \times \dfrac{5}{18} = 30$ m/sec.

Time taken to cross = 30 seconds.
Therefore, distance traveled = $30 \times 30 = 900$ m.
Distance = Length of the train + length of the bridge
900 = Length of the train + 600,
Length of the train = 300 m

4. Boats and Streams

(4a) Definitions

If a boat is moving along with the direction of the stream, then it is said to go downstream. The net speed of boat in this case is called downstream speed. A boat is moving against the direction of the stream then it is said to go upstream. The net speed of boat in this case is called upstream speed.

Let the speed of the boat in still water is 'a' km/hr and the speed of the stream is 'b' km/hr. If the boat goes downstream, then the speed will be (a + b) km/hr as in this case, the water will take the boat along with it.

If the boat goes upstream, then the speed will be (a – b) km/hr as in this case, the water will offer resistance to the boat. Let the downstream speed = D = a + b ...(i)

Then the upstream speed = U = a – b ...(ii)

Adding the two equations, we get 2a = D + U.

$\Rightarrow a = \frac{(D+U)}{2}$ which gives the speed of the boat in terms of downstream and upstream speed.Subtracting the equation (i) and (ii), we get D – U = 2b

$\Rightarrow b = \frac{(D-U)}{2}$ which gives the speed of the stream

Solved Examples:

1. A man can row 15 km/hr in still water. If the speed of current is 5 km/hr and it takes 30 hours to a man to row a place and back, then how far is the place?

Ans. The speed of downstream = 15 + 5 = 20 km/hr and the speed of upstream = 15 – 5 = 10 km/hr

Let the distance is 'x' km.

We have $\left(\frac{x}{20}\right)+\left(\frac{x}{10}\right)=30$.

Solving this equation, we get x = 200 km

2. A boat takes 1 hour less to travel 120 km downstream than to travel 96 km upstream. If speed of the current is 4 km/hr, find the speed of the boat in still water.

Ans. Let the speed of boat in still water be 'x' km/hr.

We have 96/(x – 4) – 120/(x + 4) = 1

$\Rightarrow 96(x + 4) - 120/(x - 4) = x^2 - 16$

$\Rightarrow 96x + 384 - 120x + 480 = x^2 - 16$

$\Rightarrow 864 - 24x = x^2 - 16$

$\Rightarrow x^2 + 24x - 880 = 0$

$\Rightarrow x^2 + 44x - 20x - 880 = 0$

$\Rightarrow x (x + 44) - 20 (x + 44) = 0$

$\Rightarrow (x + 44) (x - 20) = 0$

$\Rightarrow$ x = 20 km/hr (Rejecting the negative value)

3. If Anshul rows 40 km upstream and 120 km downstream taking 4 hours each time, then speed of the stream is

Ans. *Basic Formula:* Speed of the stream = $\frac{1}{2}$ (D - U)

where D = speed downstream, U = speed upstream

Speed downstream = distance travelled / time taken

$= \frac{120}{4} \Rightarrow D = 30\text{km / hr}$

Speed upstream = $\frac{40}{4} = 10 \Rightarrow U = 10$ km/hr

Speed of the stream = $\frac{1}{2}$ (D – U) km/hr

$= \frac{1}{2}(30 - 10) = 10$ km/hr

4. A man rows 750 m in 675 seconds against the stream and returns in $7\frac{1}{2}$ minutes. What is the speed of boat in still water?

Ans. Speed in still water = $\frac{1}{2}$ (D + U) km/h where 'D' is speed downstream and 'U' is speed upstream.

Speed upstream 'U' = 750/675 = 10/9 m/sec, Speed downstream 'D' = $\frac{750}{450} = \frac{5}{3}$ m/s

Speed of boat in still water = $\frac{1}{2}$ (D + U) = $\frac{1}{2}$ $\left(\frac{10}{9}+\frac{5}{3}\right)\times\frac{18}{5} = 5$ km/hr

5. Races

A race is a competition in which the competitors have to cover a given distance in the minimum possible time. The ground or path on which a race is run is called a racecourse. The point from where a race starts is called the starting point, and the point where race finishes is the winning point or goal.

As Races and Games is a sub-topic of Time, Speed and Distance, so all the formulas of Time, Speed and Distance are applicable to questions of Races and Games.

(5a) Some terminologies used in Races

- **P gives Q a start of x meters:** This statement implies that, while P starts the race from starting point, whereas, Q starts x meters ahead of P. To cover a race of 100 meters in this case, P will have to cover 100 meters while Q will have to cover only (100 – x) meters.

- **P beats Q by x meters:** This statement implies that in the same time, while P reached the winning point, whereas, Q is behind P by x m. To cover a race of 100 meters in this case, P has covered 100 meters while Q has covered only (100 – x) meters.
- **P can give Q a start of t minutes:** This statement implies that P will start t minutes after Q starts from the starting point. Both P and Q will reach the finishing point at the same time.
- **P can give Q x meters or t seconds:** This statement implies that P can give Q either x meters or t seconds from where you can find the speed of Q as x/t meters per second.
- **Dead Heat:** A dead heat situation arises when all participants reach the finishing point at the same time.

(5b) Circular Tracks

Circular tracks may involve two or three participants. They usually start from the same point and run in the same or opposite directions. Let the length of the track by 'd' metres and if radius of circle is 'r'(given) then length of the track will be $2\pi r$.

a) When two persons start from the same point and run in opposite directions with speeds 'x' and 'y' m/s, the following rules apply:

Their relative speed is always equal to sum of their individual speeds.

Between any two meetings, the total distance covered by them together is always equal to the length of the circular track.

Time taken for the first meeting after starting $= \dfrac{d}{(x+y)}$

Time taken for them to meet at the starting point for the first time = LCM of $\left[\left(\dfrac{d}{(x)}\right) and \left(\dfrac{d}{y}\right)\right]$

b) When two persons start from the same point and run in the same direction with speeds 'x' and 'y' m/s (assume x > y), the following rules apply:

Their relative speed is always equal to the difference of their individual speeds.

Between any two meetings, the faster contestant covers an extra distance equal to the length of the circular track compared to the slower contestant.

Time taken for the first meeting after starting $= \dfrac{d}{(x-y)}$

Time taken for them to meet at the starting point for the first time = LCM of $\left[\left(\dfrac{d}{(x)}\right) \text{and} \left(\dfrac{d}{y}\right)\right]$.

Solved Examples:

1. In a 400 meter race, A can give B a start of 370 meter and C a start of 333 meter. What start can B give C?

Ans. When A runs 400 meter, B runs 370 meter.
When A runs 400 meter, C runs 333 meter. i.e.
when B runs 370 meter, C runs 333 meter.
Therefore, when B runs 400 meter,
C runs $\dfrac{333\times400}{370} = 360$ meter.
Or B can give C a start of 40 meter in a 400 meter race.

2. A runs 5 times as fast as B. If A gives B a start of 100 m, how far must the destination be so that A and B reach it at the same time?

Ans. Ratio of the speeds of A and B = 5:1.
In a race of 5 m, A gains 4 m over B. 100 meters will be gained by A in race of $(\frac{5}{4}\times 100 = 125$ m).
Thus, destination must be at a distance of 125 m from the starting point.

3. There is a track with a length of 100 meters and 2 people, A & B, are running around it at 25 m/min and 10 m/min respectively in the same direction. When will A and B meet at the starting point for the first time?

Ans. The time of their meeting again at the starting point will be the LCM of $\frac{100}{25}$ & $\frac{100}{10}$ i.e. 4 & 10 LCM of 4 and 10 = 20.
So, after 20 minutes these people will be together at the starting point.

4. There is a track with a length of 200 meters and 2 people, A & B, are running around it at 15 m/min and 25 m/min respectively in the same direction. When will B overtake A for the 1st time?

Ans. Time $= \dfrac{d}{(b-a)}$.

Time of meeting $= \dfrac{200}{(25-15)} = \dfrac{200}{10} = 20$ min.

11 Alligation & Mixture

Alligation and Mixtures

Whenever we have two quantities and using these two quantities we need to create a mixture of some desired composition, then the ratio in which the two quantities are mixed is obtained by the rule of allegation.
Let us understand it with an Examples.

Solved Examples:

1. Let us say that we have two types of rice. The cost price of the first type is Rs 30 and that of the second type is Rs 70. We want to mix the two verities in such a way that the cost of the resultant mixture is Rs 45.
Now the question arises that in what ratio the two verities should be mixed?
The answer lies in the concept of weighted average. This ratio can be found by the method of the weighted average.

Ans. Let us assume that the rice of first type are taken 'm' kg and that of the second type are taken 'n' kg and are mixed and the resultant mixture is worth Rs 45/kg. We need to find the ratio of m and n.
By the concept of weighted average,

we have $\dfrac{30m+70n}{m+n}=45$

$\Rightarrow 30m + 70n = 45m + 45n$

$\Rightarrow 45m - 30m = 70n - 45n$

$\Rightarrow 15m = 25n \Rightarrow m/n = \dfrac{25}{15} = \dfrac{5}{3}$

Therefore, the two mixtures should be mixed in 5: 3 ratio.
Now, when we use the rule of allegation to get the above ratio, then we do not need such calculations. The rule states, that "When different quantities of different items are mixed together to produce a mixture having mean value, then the ratio of their quantities is inversely proportional to the differences in their values from the mean value."

To understand the above definition, write down the two different cost prices (extreme values) and write the required value in the middle. After that take the differences of the middle value from the two extreme value as shown in the figure below.

I		II
30		70
	45	
25		15
5		3

Here the difference of 70 and 45 is 25 and it becomes the part of I in the ratio. Similarly, the difference of 45 and 30 is 15 and it becomes the part of II in the ratio. The required ratio becomes 5: 3.

2. A milkman professes to sell his milk at cost price but he mixes it with water and thereby makes a profit of 20%.
What is the percentage of water in the mixture?

Ans. Let cost price of 1 litre milk be Rs. 1 and selling price of 1 litre of mixture = Rs. 1,
Also profit = 20%. Cost price of 1 litre mixture

$$= \text{Rs. } \left(\frac{100}{120}\times 1\right) = \frac{5}{6}$$

By the rule of alligation, we have:

C.P. of 1 litre of milk		C.P. of 1 litre of water
Rs. 1		0
	5/6	
5/6		1/6

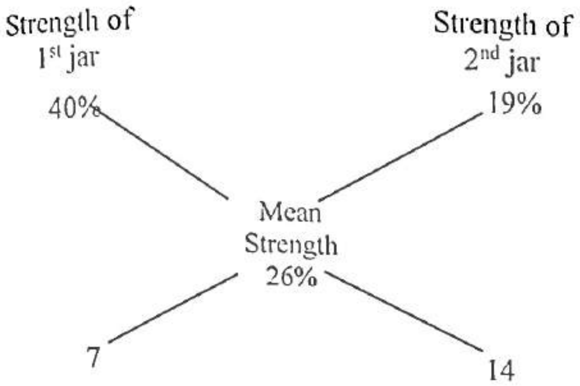
Ratio of milk to water $= \dfrac{5}{6} : \dfrac{1}{6} = 5:1$.

Hence, percentage of water in the mixture

$$= \left(\frac{1}{6}\times 100\right)\% = 16.67\%$$

3. A jar containing a mixture contains 40% milk. A part of this milk is replaced by another containing 19% milk and now the percentage of milk was found to be 26%. Find the quantity of milk replaced.

Ans. By the rule of allegation, we have:



So, the required ratio is 7: 14 = 1 : 2.

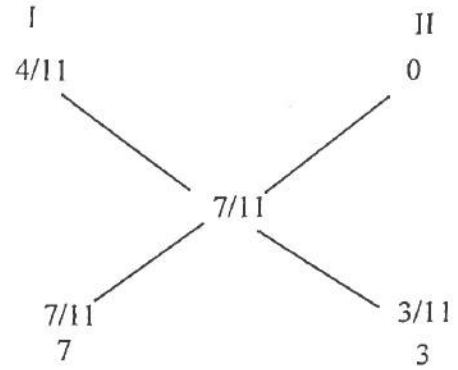
$\therefore$Required quantity replaced = $\frac{2}{3}$

4. The ratio of milk and water in one container is 4: 7. How much mixture should be replaced with pure water so that the ratio of milk and water becomes 7: 4?

Ans. The amount of milk in the container = $\frac{4}{11}$.

As we are adding the pure water, so the amount of milk added = 0, the amount of milk in the final mixture = $\frac{7}{11}$.

By the rule of allegation, we have



Here in the ratio 7: 3, 7 is representing the replaced part and 3 is representing the non-replaced part.

It means that if total parts are 10 (7 + 3), then 3 parts are replaced. Therefore, $\frac{3}{10}th$ of the total mixture should be replaced with pure water.

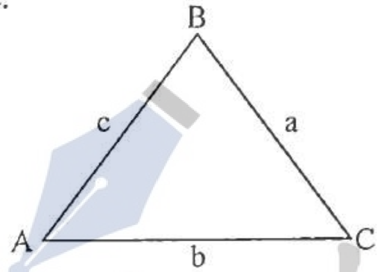
12 Mensuration (2d & 3d)

Mensuration (Two - Dimensional)

Mensuration is the branch of geometry that is concerned with the measurement of lengths, areas, volumes etc. of two-dimensional (2D) and three-dimensional (3D) figures. Let us discuss some 2D figures.

1. Triangle

Three non-parallel lines (sides) intersecting at three different points (vertices) in a plane is known as a triangle. From the figure, let, BC = a, AC = b and AB = c of triangle ABC.



(1a) Properties of Triangle

a. Sum of angles of triangle ABC $= \angle A + \angle B + \angle C = 180°$

b. Perimeter of triangle = a + b + c

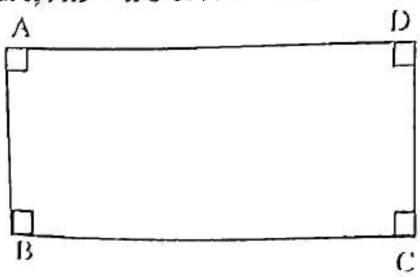
c. Area of triangle $= \frac{1}{2} \times$ Base $\times$ Height

d. Also, area of triangle $= \sqrt{[s(s-a)(s-b)(s-c)]}$, where, $s = \frac{a+b+c}{2}$ and a, b and c are length of sides of triangle

e. Area of equilateral triangle $= \frac{\sqrt{3}}{4}a^2$, where a is the side of triangle

2. Rectangle

A four-sided close figure making four right angles in a plane is known as rectangle.
From the figure, AD = BC & AB = CD



(2a) Properties of Rectangle

a. Sum of angles of rectangle ABCD $= \angle A + \angle B + \angle C + \angle D = 360°$

b. Pair of opposite sides are equal and parallel

c. All four angles are 90°

d. Diagonals are equal in length

e. Diagonals bisect each other but not at 90°

f. Diagonals do not bisect angles of a rectangle

g. Line segments joining the midpoint of sides of a rectangle forms a rhombus

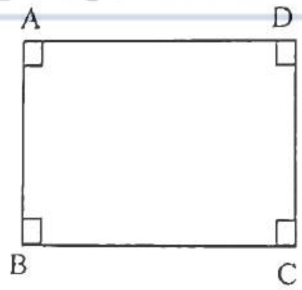
h. Perimeter of rectangle = 2(Length + Breadth)

i. Area of rectangle = Length $\times$ Breadth

j. Diagonal of rectangle $= \sqrt{(\text{Length}^2 + \text{Breadth}^2)}$

3. Square

A four-sided close figure in a plane with all sides and all angles equal is known as square.
From the figure, AB = BC = CD = DA



(3a) Properties of Square

a. Sum of angles of rectangle ABCD $= \angle A + \angle B + \angle C + \angle D = 360°$

b. Pair of opposite sides are parallel and all sides are equal

c. All angles are 90°

d. Diagonals are equal & bisect each other at 90°

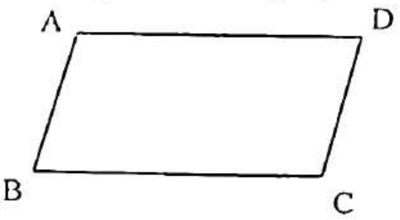
e. Diagonals are angle bisectors

f. Area of square = Side $\times$ Side & Perimeter of square = 4 $\times$ Side

4. Parallelogram

A four-sided close figure with both pairs of opposite sides is parallel is known as parallelogram.

From the figure, AB || DC and AD || BC



(4a) Properties of Parallelogram

a. Sum of angles of rectangle ABCD

$= \angle A + \angle B + \angle C + \angle D$

$= 360°$

b. Opposite sides are parallel and equal

c. Opposite angles are equal

d. Adjacent angles are supplementary

e. Diagonals are not equal

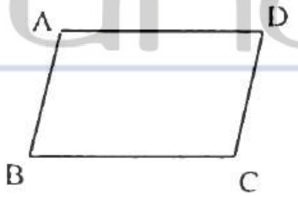
f. Diagonals bisect each other but not at 90°

g. Diagonal divides the parallelogram in two triangles of equal area

h. Area of parallelogram = Base × Height

5. Rhombus

A four-sided close figure with both pairs of opposite sides is parallel and all sides equal is known as rhombus.
From the figure, AB = BC = CD = DA



(5a) Properties of Rhombus

a. Sum of angles of rectangle ABCD

$= \angle A + \angle B + \angle C + \angle D$

$= 360°$

b. Opposite sides are parallel and all sides are equal

c. Opposite angles are equal

d. Adjacent angles are supplementary

e. Diagonals are not equal.

f. Diagonals bisect each other at 90°

g. Diagonal are angle bisectors

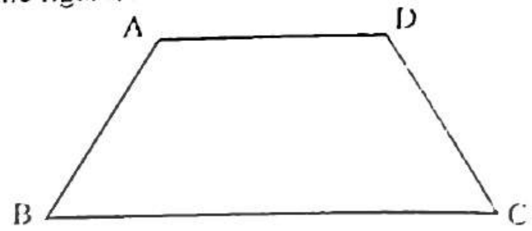
h. Line segment joining midpoints of sides forms a rectangle

i. Perimeter of rhombus = 4 × Side

j. Area of rhombus = $\frac{1}{2}$ × Product of Diagonals

6. Trapezium

A four-sided close figure in a plane with only one pair of opposite sides is parallel is known as trapezium. If length of non-parallel sides are equal then it is known as isosceles trapezium.
From the figure, AD//BC



(6a) Properties of Trapezium

a. In isosceles trapezium diagonals are equal

b. In isosceles trapezium consecutive angles along each parallel side are equal

So $\angle B = \angle C$

and $\angle A = \angle D$

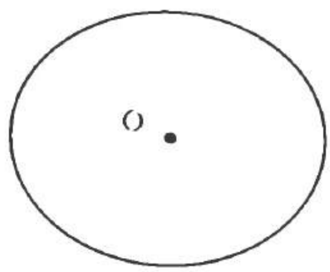
c. In isosceles trapezium pair of opposite angles are supplementary

$\angle A + \angle C = \angle B + \angle D = 180°$

d. Area of trapezium = $\frac{1}{2}$ × (Sum of Parallel Sides) × Height

7. Circle

A circle is collection of all the points which are at the same distance from a given fixed point. The fixed point is called center of the circle and fixed distance is called radius of the circle. It is a round curved shape which has no corners or edges as shown in figure.



(7a) Properties of Circle

a. Diameter of a circle = 2r, where r is the radius of the circle

b. Circumference (Perimeter) of a circle = $2\pi r$, where π is a constant = $\frac{22}{7}$ = 3.14 (approximately)

c. Area of a circle = πr^2

d. Length of arc = $\left(\frac{\theta}{360}\right)2\pi r$, where θ is angle formed in minor sector

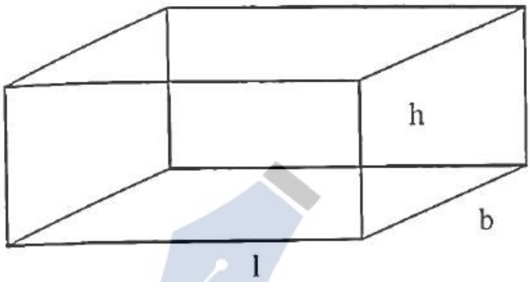
e. Area of a minor sector of a circle = $\left(\frac{\theta}{360}\right)\pi r^2$

Mensuration (Three - Dimensional)

Mensuration is the branch of geometry that is concerned with the measurement of lengths, areas, volumes etc. of two-dimensional (2D) and three-dimensional (3D) figures. Let us discuss some 3D figures.

1. Cuboid

A solid having six rectangular faces in the space at right angles to each other is known as cuboid.



(1a) Formulas related to cuboid

If length, breadth and height of cuboid are l, b, h respectively, then

Volume = $l \times b \times h$

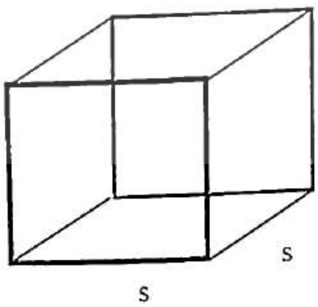
Total Surface Area = $2(lb + bh + lh)$

Lateral Surface Area or Area of 4 walls = $2(l + b)h$

Diagonal = $\sqrt{(l^2 + b^2 + h^2)}$

2. Cube

A solid having six square faces in the space at right angles to each other is known as cube.



(2a) Formulas related to cube

If side of cube is s, then

Volume = s^3

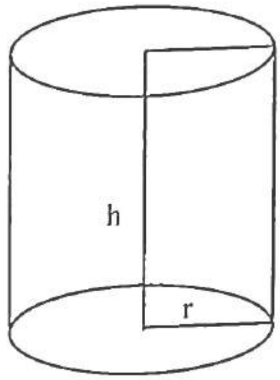
Total Surface Area = $6s^2$

Lateral Surface Area = $4s^2$

Diagonal = $s\sqrt{3}$

3. Right Circular Cylinder

Place a circular coin on the plane and keep placing more coins on the top of it, the figure thus form will be a cylinder with some height in the space. The solid enclosed by two circular planes perpendicular to the line joining the centers is known as cylinder.



(3a) Formulas related to right circular cylinder

If radius of the base of a cylinder is r and height of the cylinder is h, then

Volume = $\pi r^2 h$

Curved Surface Area = $2\pi rh$

Total Surface Area = $2\pi r(h + r)$

If the cylinder is hollow and if it's internal radius and outer radius are r and R respectively, then

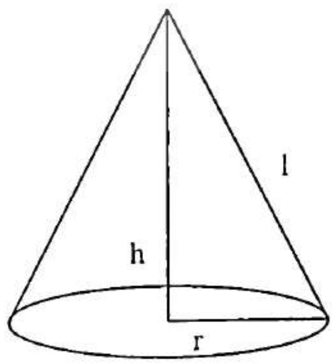
Volume = $\pi(R^2 - r^2)h$

Curved Surface Area = $2\pi(R - r)h$

Total Surface Area = $2\pi rh + 2\pi Rh + 2\pi(R^2 - r^2)$

4. Right Circular Cone

A right circular cone is a figure in which the axis of the cone is the line meeting the vertex to the centre of the circular base.



(4a) Formulas related to right circular cone

If radius of the base of a cone is r, height is h and slant height is l, then

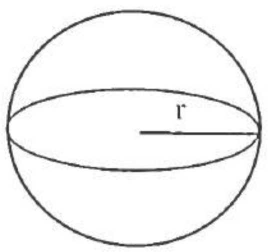
Volume $= \frac{1}{3}\pi r^2 h$;

Curved Surface Area $= \pi r l$;

Total Surface Area $= \pi r (l + r)$

5. Sphere

The set of all points in 3D space lying at the same distance (the radius) from a given point (the centre) is known as sphere. It is also the result of rotating a circle about one of its diameters.



(5a) Formulas related to Sphere

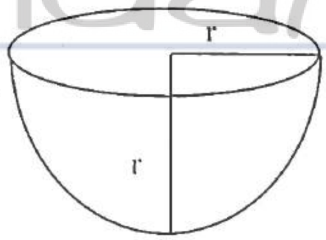
If radius of sphere is r, then

Volume $= \frac{4}{3}\pi r^3$;

Surface Area $= 4\pi r^2$

6. Hemi Sphere

Hemisphere is half of a sphere.



(6a) Formulas related to hemi sphere

If r is the radius of hemi sphere, then

Volume $= \frac{2}{3}\pi r^3$

Total Surface Area $= 3\pi r^2$

Curved Surface Area $= 2\pi r^2$

Solved Examples:

1. If length, breadth and height of a cuboidal box are in the ratio 6:5:4 and its volume is 960 cm^3, find the total surface area of the cuboidal box.

Ans. Let length, breadth and height of cuboidal box is 6a, 5a and 4a respectively.
Volume of cuboidal box $= l \times b \times h$
$960 = 6a \times 5a \times 4a \Rightarrow a = 2$
Therefore, length of cuboidal box = 6a = 12 cm
Breadth of cuboidal box = 5a = 10 cm
Height of cuboidal box = 4a = 8 cm
Total Surface Area
$= 2[(12 \times 10) + (10 \times 8) + (12 \times 8)] = 592\ cm^2$

2. A cubical box has edge 12 cm and other cuboidal box has dimensions $13.5 \times 10 \times 8$. Which box has the greater lateral surface area?

Ans. Side of cube = s = 12 cm
Lateral surface area of cube
$= 4s^2 = 4 \times 12 \times 12 = 576\ cm^2$
For cuboid, length = 13.5 cm,
breadth = 10 cm and height = 8 cm
Lateral surface area of cuboid
$= 2(l + b)h = 2(13.5 + 10) \times 8 = 376\ cm^2$
So, cube has a greater lateral surface area

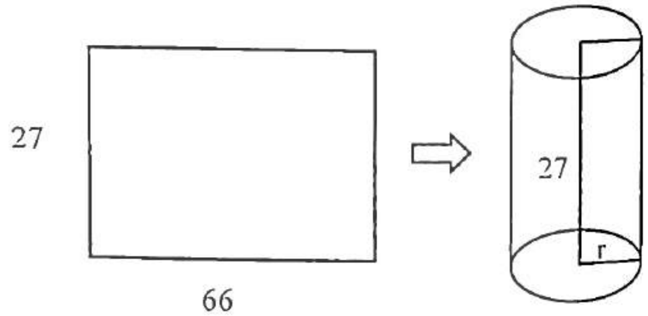
3. A cube of side 10 cm is immersed completely in a rectangular vessel containing water. If the dimensions of base of cuboidal vessel are 8×5, find the rise in water level in the vessel.

Ans. Side of cube = 10 cm
Volume of cube $= side^3 = 10^3 = 1000\ cm^3$
If the cube is immersed in the vessel the water level rises in the vessel.

Let the rise in water level = h cm
Therefore, volume of the cube
= Volume of the water displaced by the cube
$1000 = 8 \times 5 \times h \Rightarrow h = 25$ cm

4. A rectangular sheet of paper 66 cm × 27 cm is rolled along its length and cylinder is formed. Find the radius of the cylinder.

Ans. When the rectangular sheet is rolled along its length, we find that the length of the sheet forms the circumference of the base of the cylinder and breadth becomes the height of the cylinder.



Let radius of cylinder formed = r cm and
circumference of base $= 2\pi r = 66$

$\Rightarrow \qquad r = \frac{21}{2}$ cm

5. A pipe 40 cm long has interior diameter 14 cm. If the thickness of the pipe is 7 cm. Find its total surface area.

Ans. Interior diameter = 14 cm
$\Rightarrow$ interior radius = r = 7 cm

As thickness of pipe is 7 cm,
therefore outer radius = R = 7 + 7 = 14 cm
Height of pipe = h = 40 cm So total surface area of pipe = $2\pi Rh + 2\pi rh + 2\pi (R^2 - r^2)$

$$= 2 \times \frac{22}{7} \times 14 \times 40 + 2 \times \frac{22}{7} \times 7 \times 40 + 2 \times \frac{22}{7} \times (14^2 - 7^2) = 6204 \text{ cm}^2$$

13 Geometry

Geometry (Line, Angle and Triangle)

'Geometry' is the study of things of different shapes, how these shapes are related to each other and finding their properties. So, we can define Geometry as "the branch of mathematics concerned with the properties and relations of points, lines, surfaces, solids, and higher dimensional figures".

Let us start discussing the basic Geometry, which includes study of points, line, angles, surfaces and solids. Let us define these terms.

1. Point

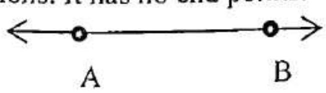
A point is defined as a mark of position which has no shape, no size. Therefore, it is a zero-dimensional figure. In geometry we generally represent a point by a capital letter.

(1a) Properties of Point:

a) Infinite lines can pass through a point
b) One and only one line can pass two distinct points
c) Infinite circles can pass through two distinct points
d) If a line passing through three distinct points, they are said to be collinear, otherwise they are non-collinear
e) One and only one circle can pass three distinct points
f) If a circle passes through four or more than four points, they are said to be concyclic

2. Line

A line is a collection of points that can extend indefinitely in both directions. It has no end points.

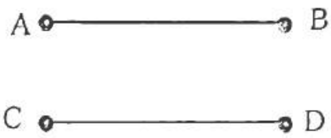


(2a) Line Segment

A line segment is the part of a line with two fixed end points.

3. Parallel Lines

Two lines in a plane are said to be parallel if perpendicular distance between them is constant throughout and they never meet when produced indefinitely on either side. If two straight lines make equal angle with same line then they are parallel.



(3a) Properties of parallel lines:

If AB || CD and EF is transversal line then

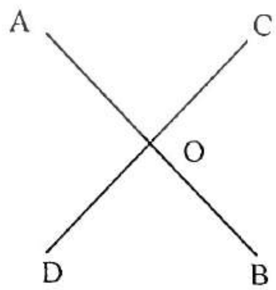
a) Corresponding angles are always equal.
From figure
$\angle 1 = \angle 2$ (corresponding angle)

b) Alternate interior angles are always equal.
From figure
$\angle 2 = \angle 3$

c) Co- interior angles are supplementary.
From figure
$\angle 2 + \angle 4 = 180°$
And $\angle 3 + \angle 5 = 180°$

4. Intersecting lines

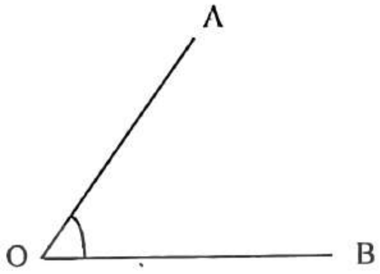
When two lines meet in a plane, they are called intersecting lines. In the figure lines AB and CD intersects each other at O.



5. Angle

An angle is defined as the figure formed by two lines or two rays meeting at a common end point. An angle is represented by the symbol $\angle$. In given figure the angle is

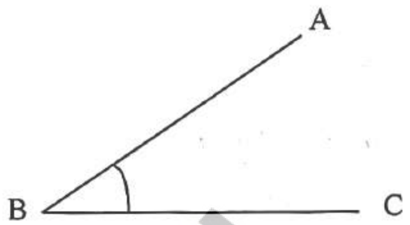
∠AOB. Angles are measured in degrees, using a protractor.



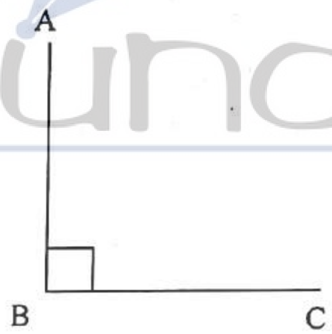
(5a) Types of Angles

1. **Acute Angle:** An angle whose measure is less than 90° is known as acute angle.

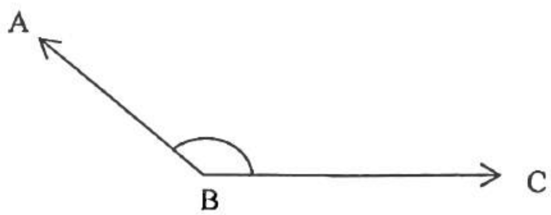
   In the figure, ∠ ABC < 90°



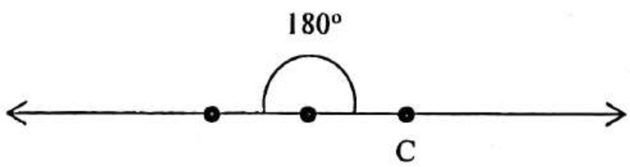
2. **Right angle:** An angle whose measure is 90° is known as right angle. In the figure, ∠ABC = 90°



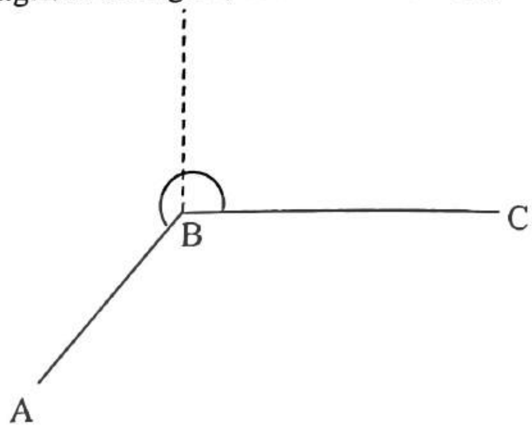
3. **Obtuse Angle:** An angle whose measure is more than 90° but less than 180° is known as obtuse angle. In the figure, 90° < ∠ ABC < 180°



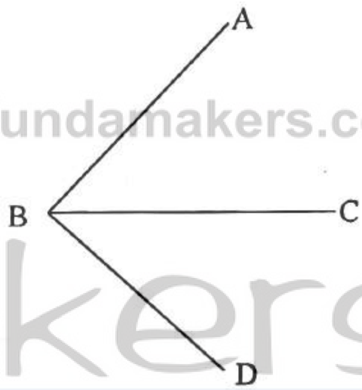
4. **Straight Line Angle:** An angle whose measure is 180°. In the figure, ∠ABC = 180°



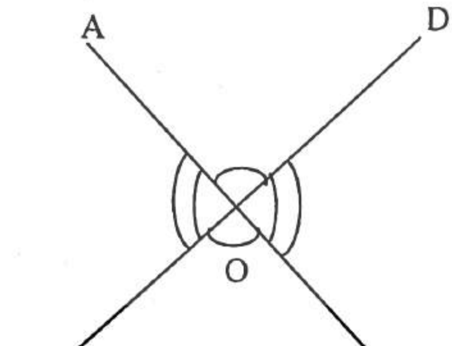
5. **Reflex Angle:** An angle whose measure is more than 180 ° but less than 360° is known as reflex angle. In the figure, 180° < ∠ ABC < 360°



6. **Adjacent Angles:** Two angles are said to be adjacent, if they have one arm in common and a common vertex. In figure ∠ABC and ∠DBC are adjacent angles with BC as a common arm and B as a common vertex.



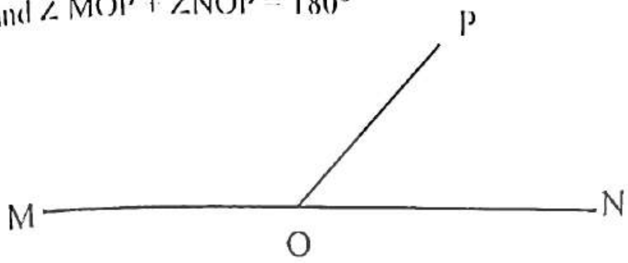
7. **Vertically Opposite Angles:** The angles opposite to the common vertex formed by the intersection of two lines are known as vertically opposite angles. In figure ∠ AOD and ∠BOC are vertically opposite angles. Also, ∠ AOD = ∠BOC



8. **Complementary Angles:** Two angles are known as complementary angles, if their sum is 90°

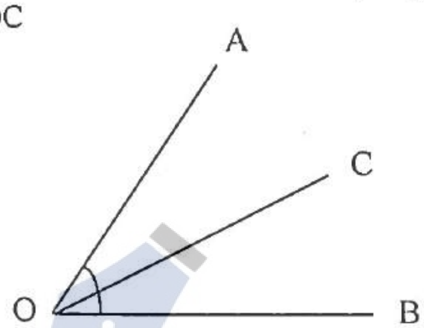
9. **Supplementary Angles:** Two angles are known as supplementary angles, if their sum is 180°

10. **Linear Pair:** A pair of adjacent angles is said to form a linear pair if the outer arms of the angles lie on same line. A linear pair is measured as 180°. In figure $\angle$ MOP and $\angle$NOP together forms linear pair and $\angle$ MOP + $\angle$NOP = 180°



6. Angle Bisector

Angle bisector divides the given angle in two equal parts. In figure line OC bisects $\angle$AOB in two equal parts. i.e., $\angle$ AOC = $\angle$ BOC



7. Triangle

A triangle is a polygon with three edges, three angles and three vertices. It is a two-dimensional figure.

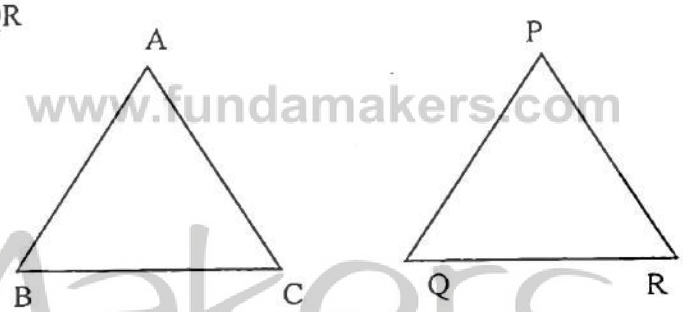
(7a) Types of Triangles

Types of triangles can be discussed according to two factors

1. **According to Sides:**
   a) **Scalene Triangle:** A triangle in which all three sides are unequal is known as scalene triangle.
   b) **Isosceles triangle:** A triangle in which any two sides are equal is known as isosceles triangle. As two sides are equal, therefore, angles opposite to equal sides are also equal.
   c) **Equilateral Triangle:** A triangle in which all three sides and all three angles are equal is known as equilateral triangle.
2. **According to Angles:**
   a) **Acute Angled Triangle:** If all the three angles of a triangle are acute angles, then it is known an acute angled triangle. In acute angle triangle sum of any two angles is always greater than 90°. In acute angle triangle $c^2 < a^2 + b^2$, where a, b & c are length of sides of triangle and c is greatest side.
   b) **Right angled Triangle:** In triangle, if measure of one angle is 90°, then it is known as right angle triangle or if sum of two angles is equal to third angle than triangle is right angled triangle. In right angle triangle $c^2 = a^2 + b^2$, where a, b & c are length of sides of triangle and c is greatest side.
   c) **Obtuse angled Triangle:** If one of the angles of triangle is obtuse angle than it is known as obtuse angled triangle or if sum of two angles is less than 90°, than triangle is obtuse angled triangle. In obtuse angle triangle $c^2 > a^2 + b^2$, where a, b & c are length of sides of triangle and c is greatest side.

(7b) Similarity of Triangles

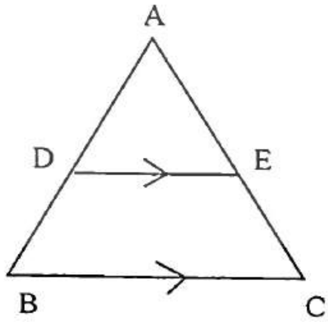
Two triangles are similar if they are equal in shape but not necessarily equal in size. Or we can say two triangles are similar if their corresponding angles are equal and their corresponding sides are proportional. In ΔABC and Δ PQR



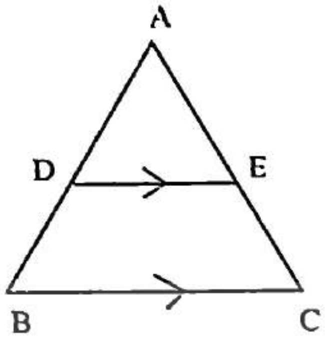
If $\angle$ A = $\angle$ P, $\angle$ B = $\angle$ Q, $\angle$ C = $\angle$ R and

$\dfrac{AB}{PQ} = \dfrac{BC}{QR} = \dfrac{AC}{PR}$ then $\Delta ABC \sim \Delta PQR$

a) **Basic proportionality theorem or Thales theorem:** If a line is drawn parallel to one side of a triangle intersecting the other two sides, then it divides the two sides in the same ratio. If DE||BC then $\dfrac{AD}{BD} = \dfrac{AE}{CE}$



b) **Mid-Point theorem:** The line joining the midpoint of two sides of a triangle is parallel and equal to half of third side.
Therefore, if AD = BD and AE = CE then DE||BC and $DE = \dfrac{1}{2}BC$



(7c) General properties related to triangle

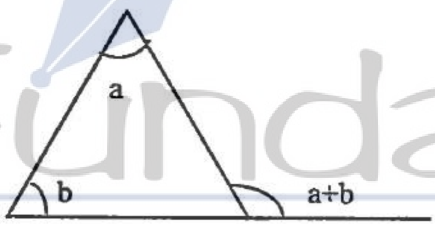
a) The sum of the three angles of a triangle is 180°

b) Sum of any two sides of a triangle is greater than the third side.

c) Difference between any two sides of a triangle is less than the third side.

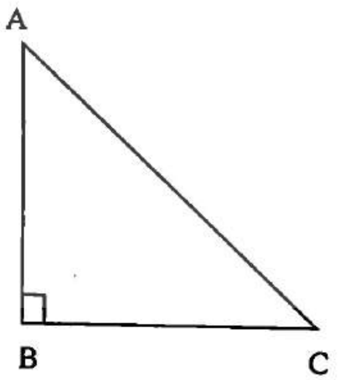
d) If two sides of triangle are unequal, the longer side has greater angle opposite to it.

e) If a side of triangle is produced the exterior angle so formed is equal to the sum of the two interior opposite angles.



f) **Pythagoras Theorem:** In right angle triangle right angled at B, $AB^2 + BC^2 = AC^2$

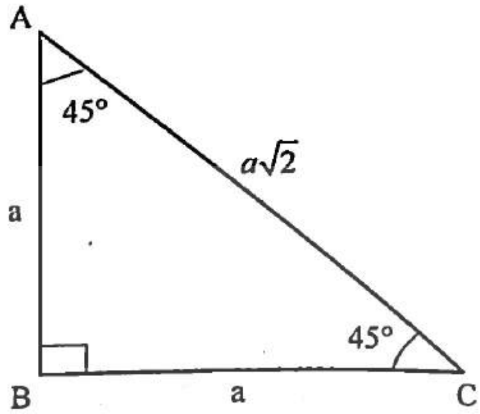
Some Pythagoras triplets – (3, 4, 5); (5, 12, 13); (7, 24, 25); (8, 15, 17); (20, 21, 29) etc.



g) **Special types of right-angled triangles**

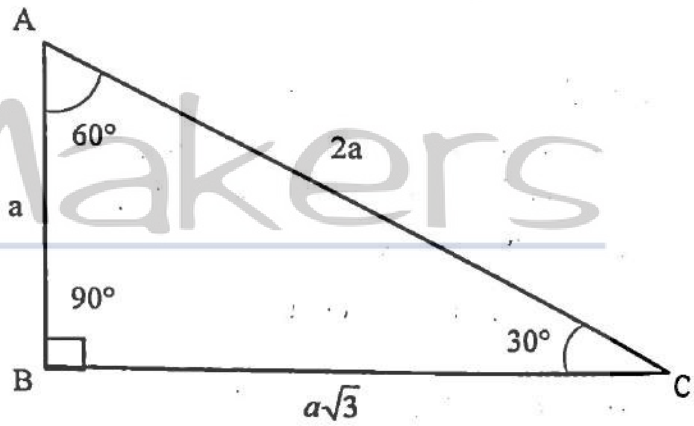
<u>**45°-45°-90°**</u>

- One of the interior angles is right angle (90°) and the two sides are equal.
  $\angle B = 90°, \angle A = \angle C = 45°$
  $AB = AC = a, AC = a\sqrt{2}$
- Perimeter = $2a + a\sqrt{2}$
- Area = $1/2 \times a^2$



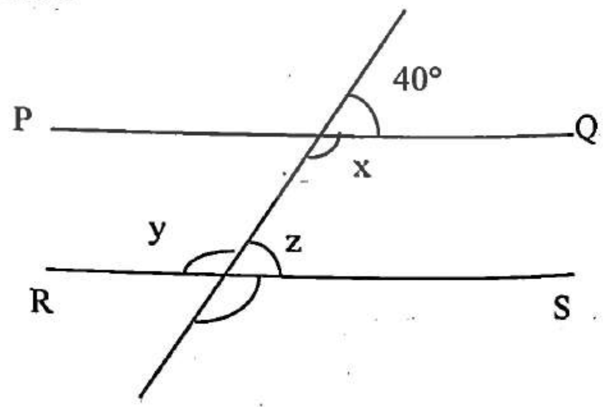
<u>**30°-60°-90°**</u>

- $\angle B = 90°, \angle A = 60°, \angle C = 30°$
- $AB = a, BC = a\sqrt{3}, AC = 2a$
- Perimeter = $3a + a\sqrt{3}$
- Area = $\sqrt{3}/2\ a^2$



Solved Examples:

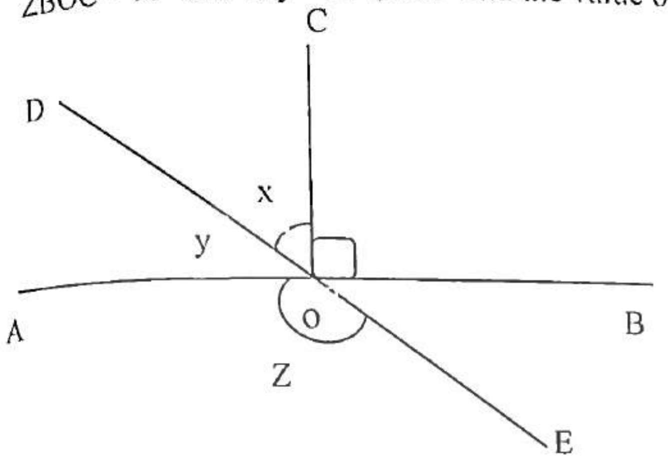
1. In the following figure PQ || RS, find values of x, y and z



Ans. From figure z = 40° (corresponding angles)
y + z = 180° (linear pair)
⇒ y = 140°

And $x = y = 140°$ (alternate interior angles)

2. In figure AB and DE intersect each other at O. If $\angle BOC = 90°$ and $x : y = 5:4$, then find the value of z.



Ans. Let $x = 5a$ and $y = 4a$

As $\angle BOC = 90°$ so $\angle AOC = 90°$ (linear pair)

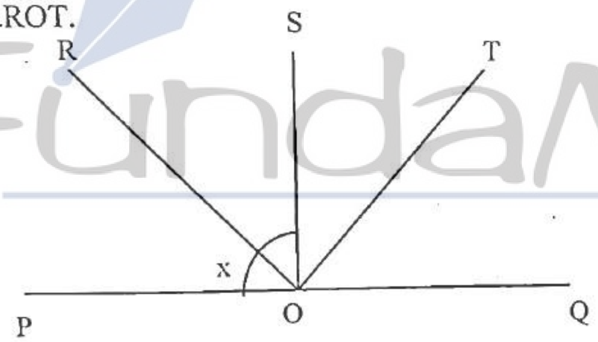
$\therefore x + y = 90° \Rightarrow 5a + 4a = 90° \Rightarrow a = 10°$.

So, we have $y = 4a = 40°$

Also, $y + z = 180°$ (linear pair)

$\Rightarrow \quad z = 180° - 40° = 140°$

3. In figure, if $\angle POS = x$ and OR and OT are angle bisectors of $\angle POS$ and $\angle SOQ$ respectively. Find $\angle ROT$.



Ans. As $\angle POS = x$, so $\angle QOS = 180° - x$

Now as OR and OT are angle bisectors of $\angle POS$ and $\angle SOQ$ respectively therefore $\angle ROS = \frac{x}{2}$ and

$\angle SOT = \frac{180° - x}{2}$.

Hence $\angle ROT = \angle ROS + \angle SOT$

$$= \frac{x}{2} + \frac{180° - x}{2} = 90°.$$

Geometry (Quadrilateral, Polygon and Circle)

1. Types of Quadrilaterals

(1a) Basic properties of Quadrilateral

a. Sum of interior angles of quadrilateral is 360°.

b. Sum of exterior angles of quadrilateral is 360°.

c. Area of quadrilateral joining midpoints of sides is equal to half of original quadrilateral.

d. Quadrilateral formed by joining midpoints of sides of original quadrilateral is a parallelogram.

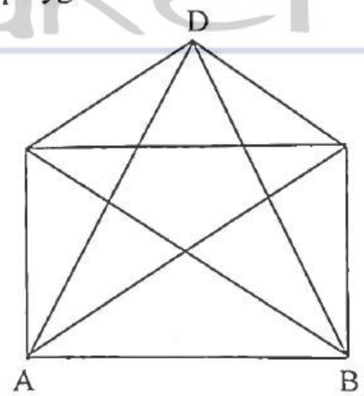
2. Polygon

A plane geometrical figure, bounded by at least three-line segments, is known as polygon. The name of polygon depends upon the number of sides it has. This can be easily understood by table given below:

Number of Sides	Name of Polygon
3	Triangle
4	Quadrilateral
5	Pentagon
6	Hexagon
7	Heptagon
8	Octagon
9	Nonagon
10	Decagon

(2a) Diagonal of Polygon

Line segment joining any two non- consecutive vertices of polygon are known as diagonal of polygon as shown in the figure. In figure AD, BD, EC, AC and BE are diagonals of polygon.



(2b) Formulas Related to Polygon

Angles:

Any polygon has as an equal number of corners and sides. The angles related to polygon are of two types

1. **Interior Angle :**Sum of interior angles of a polygon $= (n - 2) \times 180°$, where n = number of sides of polygon

2. **Exterior Angle :**The exterior angle is supplement to interior angle. Therefore Interior angle + Exterior angle = 180° and sum of all exterior angles = 360°

3. **Number of Diagonals:** Number of diagonals in a polygon of 'n' sides is given by $\frac{n(n-3)}{2}$

(2c) Formulas for Regular Polygon

1. Each interior angle $= \dfrac{(n-2)\times 180}{n}$,

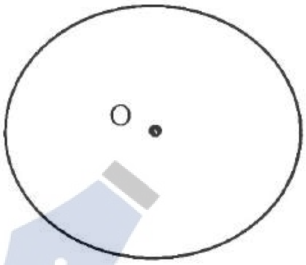
   where n = number of sides of polygon

2. Exterior angle $= \dfrac{360^\circ}{n}$,

   where n = number of sides of polygon

3. Circle

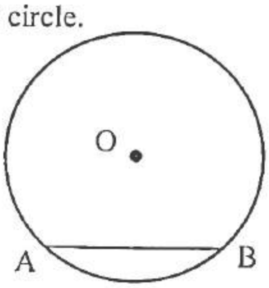
A circle can be defined as closed, two-dimensional round shape which has no corners or edges. We can say circle is a simple closed curve, all the points of which are at same distance from a given fixed point. This fixed point is called centre of the circle and fixed distance is called radius of the circle.



(3a) Definitions related to circle

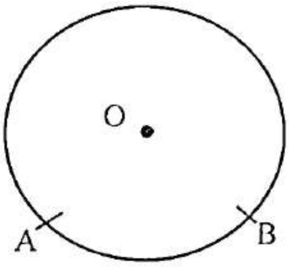
1. **Radius:** Length of line joining centre of circle to any point on the circle is known as radius of circle. In figure OA is the radius of the circle.

2. **Chord:** A line segment joining any two points on the circle is called chord of the circle. In figure AB is the chord of the circle.

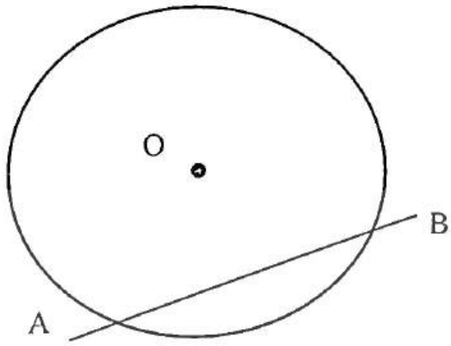


3. **Diameter:** A chord passing through the center of the circle is known as diameter of the circle. Diameter is the longest chord of the circle. In figure PQ is the diameter of the circle.

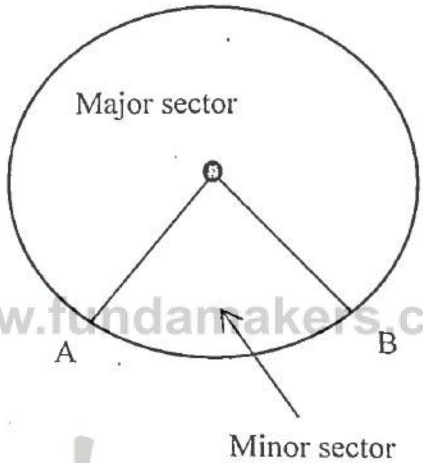
4. **Arc:** A continuous piece of circle is known as arc of the circle. In figure AB is the arc of the circle.



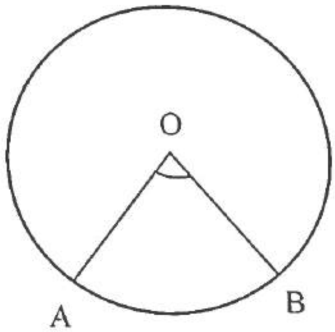
5. **Secant:** A straight line intersecting cirle at two points is known as secant of the circle. In figure AB is the secant of the circle.



6. **Sector:** The part of a circle enclosed by an arc and two radii is known as sector.

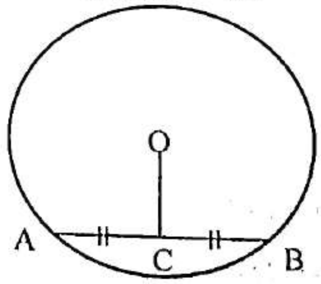


7. **Central Angle:** An angle subtended by an arc at the centre is called a central angle. In figure ∠AOB is known as central angle.



(3b) Properties of Circle

a. The perpendicular from centre of a circle to a chord bisects the chord.
   If $OC \perp AB$ then $AC = BC$



b. The line joining centre of a circle to the midpoint of a chord is perpendicular to the chord.

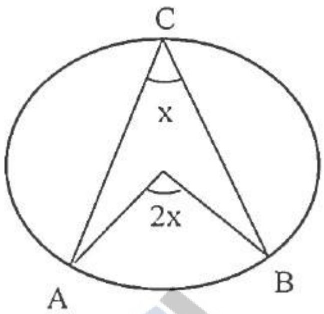
c. Perpendicular bisector of a chord always passes through the centre.

d. There is one and only one circle passing through three non collinear points.

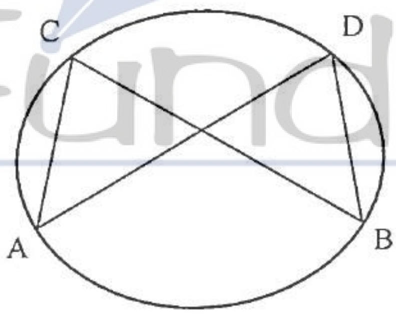
e. If diameter of a circle bisects each of the two chords of a circle, then the chords are parallel.

f. Equal chords are equidistant from the centre.

g. The angle subtended by an arc at the centre is twice the angle subtended by it at any point on the remaining part of the circle. $\angle AOB = 2\angle ACB$

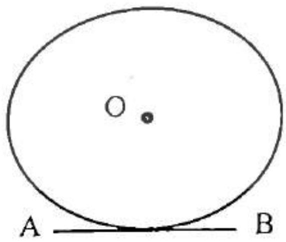


h. Angles in same segment are always equal. $\angle ACB = \angle ADB$



(3c) Tangent to a circle

A straight line touching circle at one and only one point is known as tangent of the circle. In figure AB is the tangent of the circle.



Solved Examples:

1. Find the number of sides of a polygon, the sum of whose interior angle is 1440°

Ans. As sum of interior angles of polygon
$= (n-2) \times 180°$ where,
n = number of sides of polygon,
So, we have $1440° = (n-2) \times 180° \Rightarrow n = 10$

2. Find each interior angle of a regular polygon having 9 sides.

Ans. As Interior angle of regular polygon

$$= \frac{(n-2)\times 180}{n},$$

where n = number of sides of polygon
So Interior angle

$$= \frac{(9-2)\times 180}{9} = \frac{7\times 180}{9} = 140°$$

3. Find number of diagonals of pentagon.

Ans. Number of diagonals $= \frac{n(n-3)}{2} = \frac{5\times 2}{2} = 5$

4. If the diameter of circle is 34 cm and length of one of its chords is 30 cm, then find the distance of chord from the centre.

Ans. Diameter of circle = 34 cm, therefore radius of circle = 17 c
Length of chord = 30 cm
As, perpendicular from the centre of a circle to a chord bisects the chord, so $OC \perp AB$
$\Rightarrow$ AC = 15 cm
In right ΔAOC, using Pythagoras Theorem
$OA^2 = AC^2 + OC^2$
Therefore OC = 8 cm
So, distance of chord from the centre is 8 cm.

Basic Concepts of Trigonometry

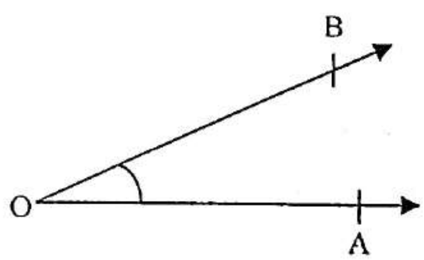
The word 'trigonometry' is derived from the Greek words 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, trigonometry is the study of relationships between the sides and angles of a triangle.

1. Definitions & Review of Concepts

(1a) Angle

When a ray OA rotates about the point O from an initial position OA to the final position OB, we say that $\angle AOB$ has been formed.

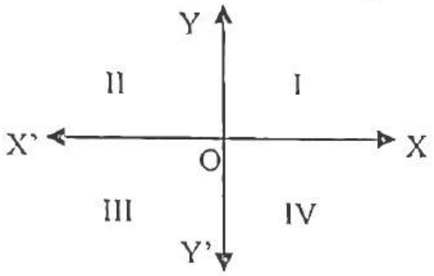
OA is called the initial side and OB is the terminal side of $\angle AOB$.



(1b) Quadrants:

Let X'OX and YOY' be two lines at right angles to each other. We call X'OX and YOY' as X-axis and Y-axis respectively. These lines divide the plane into 4 parts. The parts XOY, YOX', X'OY' and Y'OX are known as 1st, 2nd, 3rd and 4th quadrant respectively.

An angle is said to be in a particular quadrant, if the terminal side of the angle lies in that quadrant.



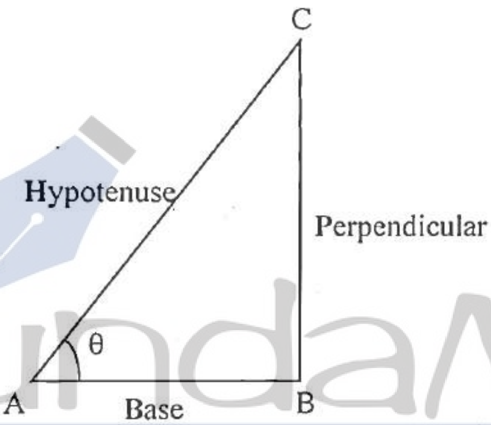
(1c) Trigonometric Ratios:

$\frac{P}{H} = \sin\theta$

$\frac{B}{H} = \cos\theta$

$\frac{P}{B} = \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\frac{H}{P} = \text{cosec}\,\theta = \frac{1}{\sin\theta}$



$$\frac{H}{B} = \sec\theta = \frac{1}{\cos\theta}$$

$$\frac{B}{P} = \cot\theta = \frac{1}{\tan\theta}$$

(1d) Trigonometric Identities and Pythagoras Theorem:

Pythagoras Theorem, $P^2 + B^2 = H^2$

Dividing by H^2 throughout,

$$\frac{P^2}{H^2} + \frac{B^2}{H^2} = \frac{H^2}{H^2} \Rightarrow \sin^2\theta + \cos^2\theta = 1$$

Dividing by B^2 throughout, $\frac{P^2}{B^2} + \frac{B^2}{B^2} = \frac{H^2}{B^2}$

$\Rightarrow \quad \tan^2\theta + 1 = \sec^2\theta$

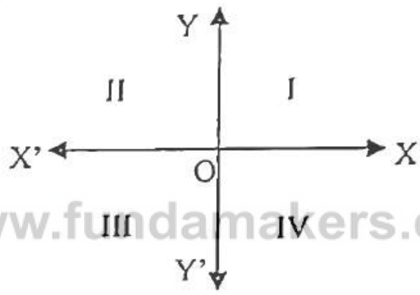
Dividing by P^2 throughout, $\frac{P^2}{P^2} + \frac{B^2}{P^2} = \frac{H^2}{P^2}$

$\Rightarrow \quad 1 + \cot^2\theta = \text{cosec}^2\theta$

(1e) Values of T-Ratios:

Angle	$\sin\theta$	$\cos\theta$	$\tan\theta$
0^0	0	1	0
$30^0 = \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45^0 = \frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$60^0 = \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90^0 = \frac{\pi}{2}$	1	0	∞
$180^0 = \pi$	0	-1	0

(1f) Sign of T-Ratios:



1st Quadrant: All positive

2nd Quadrant: $\sin\theta$ and $\text{cosec}\,\theta$ positive

3rd Quadrant: $\tan\theta$ and $\cot\theta$ positive

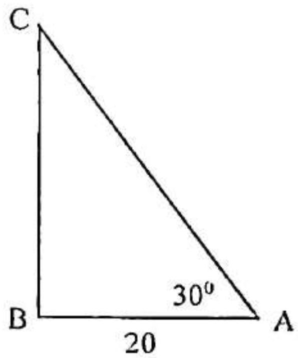
4th Quadrant: $\cos\theta$ and $\sec\theta$ positive

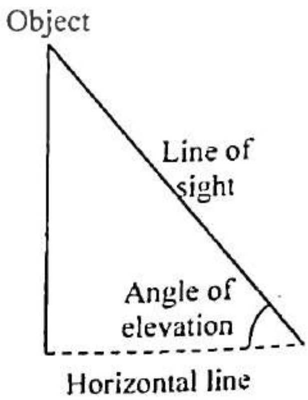
Height and Distance

1. Angles of Elevation and Depression

Suppose we wish to determine the height of a tall tree without climbing to the top of it. We could stand on the ground at a point some distance (say 20 m) from the foot B of the tree. Suppose we are able to measure angle BAC and we find it to be 30^0. Then, as shown in figure, we can calculate the height the tree BC by using,

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{BC}{20} \Rightarrow BC = \frac{20}{\sqrt{3}} = 11.5 \text{ m (approx.)}$$





Suppose we are viewing an object. The line of sight or the line of vision is a straight line from our eye to the object we are viewing. If the object is above the horizontal line of sight from the eye (i.e., at an upper level than ourselves), then we have to turn our head upwards to view the object. In the process, our eyes move through an angle. This angle is called the angle of elevation of the object.

If the object is below the horizontal line of sight from the eye (i.e., at a lower level than ourselves), then we have to turn our head downwards to view the object. In the process, our eyes move through an angle. This angle is called the angle of depression of the object.

Solved Examples:

1. A man wishes to find the height of a flag post. The angle of elevation of the top of the flag post is found to be 45^0. On walking 30 meters towards the tower he finds the corresponding angle of elevation to be 60^0. Find the height of the flag post.

(a) 62 m 2. 82 m 3. 71 m 4. $30\sqrt{3}$ m

Ans. Let AB = height of flag post = x m

In $\triangle$ ABD, $\tan 60^0 = \dfrac{x}{BD}$

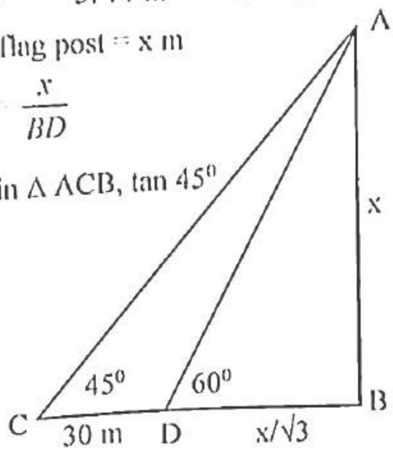
$\Rightarrow$ BD = $\dfrac{x}{\sqrt{3}}$ Also, in Δ ACB, $\tan 45^0$

$= \dfrac{AB}{BD + DC}$

$\Rightarrow \dfrac{x}{\sqrt{3}} + 30 = x$

$\Rightarrow$

$\dfrac{x(\sqrt{3}-1)}{\sqrt{3}} = 30$



$\Rightarrow x = \dfrac{30\sqrt{3}}{0.732} \approx 71\,\text{m}$

2. A small boy is standing at some distance from a flag post. When he sees the flag, the angle of elevation formed is 60^0. If the height of the flag post is 30 ft., what is the distance of the child from the flag post?

(a) $15\sqrt{3}$ ft (b) $10\sqrt{3}$ ft (c) $20\sqrt{3}$ ft (d) $\dfrac{20}{\sqrt{3}}$ ft

Ans. In ΔABC, $\dfrac{AB}{BC} = \tan 60^0$

$\Rightarrow \quad \dfrac{30}{BC} = \sqrt{3}$

$\Rightarrow \quad BC = \dfrac{\sqrt{3}\times\sqrt{3}\times 10}{\sqrt{3}}$

$\Rightarrow \quad BC = 10\sqrt{3}$ ft.

3. The angles of elevation of top and bottom of a flag kept on a flag post from 30 meters distance are 45^0 and 30^0 respectively. What is the height of the flag?

(a) 17.32 m (a) 14.32 m

(c) 12.68 m (d) $12\sqrt{3}$ m

Ans. Solving,

$\tan 45^0 = \dfrac{AC}{30}$

$\Rightarrow$ AC = 30 m

Also, $\tan 30^0$

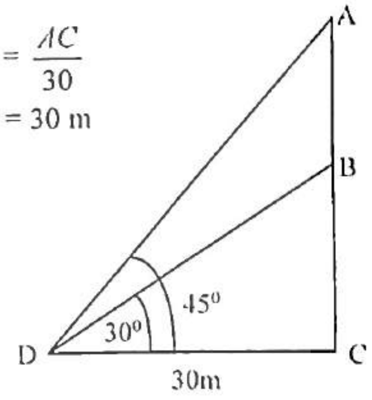
$= \dfrac{BC}{30} \Rightarrow BC = \dfrac{30}{\sqrt{3}}$

Height of flag

AB = AC − BC

$= 30 - \dfrac{30}{\sqrt{3}}$

= 12.68 m



14 Clocks & Calendars

1. Clocks

The dial of any standard clock is divided into 12 equal spaces, which stand for the hours of the day. Each of these hour spaces are next divided into 5 equal spaces – this denotes the duration of a minute. The minute hand and hour hand are 2 major hands that keep rotating in the clock.

Initially, there would be two type of questions.

1. A particular time is given and we have to calculate the angle between two hands.
2. A particular angle is given and we have to calculate the time at which the given angle is made.

(1a) Speed of the minute hand

A clock is just like a complete circle having 360°. It has been divided into 12 equal parts i.e. each part is $\frac{360}{12} = 30°$. As minute hand takes a complete round of 360° in one hour i.e. 60 minute, so we can say that it covers 360° in 60 min. In 1 min, it covers $\frac{360}{60} = 6°$. So its speed is 6° per minute.

(1b) Speed of the hour hand

As hour hand covers 360° in 12 hrs, this implies that it covers 30° in 60 min. Speed = $\frac{1}{2}$ degree per minute.

Relative Speed of Min Hand and Hour Hand $= \left(6 - \frac{1}{2}\right) =$ 5.5°/Minute (as both hands are in same direction)

(1c) Angle between minute hand and hour hand

$$\theta = \left|\frac{11}{2}M - 30H\right|$$

$$\theta = \frac{11}{2}M - 30H \text{ (if } \frac{11}{2}M > 30H)$$

$$\theta = 30H - \frac{11}{2}M \text{ (if } 30H > \frac{11}{2}M)$$

H = Position of Hour hand

M = Position of Minute hand

(1d) Important Points

1. Min. & Hr. hand coincide with each other 11 times in 12 hrs duration. It is also known as hands are in a straight line in the same direction or we can say that angle of 0°.
2. In every hour, maximum one angle of 180° can be made between the two hands of the clock, but in a 12 hr. period there are 11 such angles. It is also known as hands are in a straight line in the opposite direction.
3. In every hr, any other angle between 0° and 180°, both excluded can be made maximum two times but in a 12 hr. period there are 22 such angles.

Solved Examples:

1. Find the angle between minute hand and hour hand at 4:40.

Ans. $\theta = \left|\frac{11}{2}M - 30H\right|$

Here H = 4 and M = 40

So $\theta = \left|\frac{11}{2} \times 40 - 30 \times 4\right| = 100°$

2. Find the time between 1 and 2O' clock at which the minute hand and hour hand
   (a) Make an angle of 15° with each other.
   (b) Overlap each other.
   (c) Are in straight line but in the opposite directions.

Ans. (a) $\theta = \left|\frac{11}{2}M - 30H\right|$, Between 1 and 2O' clock,

H = 1 and Θ = 15 degree

Case 1 $\theta = \frac{11}{2}M - 30H$, $\frac{11}{2}M - 30 \times 1 = 15$

$\Rightarrow M = \frac{90}{11} = 8\frac{2}{11}, 8\frac{2}{11}$ minutes past 1

Case 2 $\theta = 30H - \frac{11}{2}M$

$30 \times 1 - \frac{11}{2}M = 15 \Rightarrow \frac{11}{2}M = 15,$

$M = \frac{30}{11} = 2\frac{8}{11}$ so time is $2\frac{8}{11}$ minutes past 1.

So the angle between two hands- minute hand and hour hand is 15° is possible first time at

$2\frac{8}{11}$ minutes past 1 and second time at $8\frac{2}{11}$ minutes past 1

(b) Two hands overlap each other when $\theta = 0^o$, here H = 1

$$\theta = \left|\frac{11}{2}M - 30H\right| \frac{11}{2}\text{M} - 30 \times 1 = 0$$

$M = \frac{60}{11} = 5\frac{5}{11}$, So both hands overlap each other at $5\frac{5}{11}$ past 1.

Case 2 is same as Case I because angle of 0° can be made maximum one time in one hour duration.

(c) Two hands are in straight line and in opposite directions when θ = 180°, here H = 1

$$\theta = \left|\frac{11}{2}M - 30H\right| \frac{11}{2}\text{M} - 30 \times 1 = 180$$

M = $\frac{420}{11} = 38\frac{2}{11}$, So both hands overlap each other at $38\frac{2}{11}$ past 1.

Case 2 is not possible because angle of 180° can be made maximum one time in one hour duration.

2. Calendars

Let us consider a situation that you have to find the day of the week on 15th August 1947, it would be a difficult for you if you do not know the proper method. This method to find the day on a particular date lies on the concept of "odd days". Odd days are the days remaining after the completion of exact number of weeks or we can say that odd days are the remainder obtained when we divide number of days by 7.

Example. In an ordinary year (non – leap year) there are a total of 365 days, which means 52 × 7 + 1, or 52 complete weeks and one more day. This additional day is called as 'odd day'.

2(a) Leap and ordinary years

In an ordinary year (non – leap year) there are a total of 365 days and in a leap year, there are 366 days, one day extra because of one extra day in the month of February and 366 days = 52 × 7 + 2, There are 52 complete weeks and 2 extra days and these extra two days are called odd days. So in leap year there are 2 odd days and in an ordinary year there is one odd day.

Any year which is multiple of 4 is a leap year like 1876, 1984, 2004, and 2020 except for the century year.

Note – In case of century year, a multiple of 400 will be a leap year like 400, 800, 1200, 1600, 2000. 100, 200, 300, 1900 are not leap years.

(2b) Calculation of Odd Days

The concept of number of odd days is needed in finding the day of the week on a given date. For example in a century(100 years) if we are interested in finding the number of odd days then we need to proceed as follows. There will be 24 leap years and 76 non–leap years. This means that there will be 24 × 2 + 76 × 1 = 124 odd days. Since 7 odd days make a week, to find out the net odd days, divide 124 by 7. The remainder is 5 and this is the number of odd days in a century.

So it is important to note the following points related to the number of odd days in the concept of calendars.

100 years give us 5 odd days as calculated above.

200 years give us 5 × 2 = 10 – 7 (one week)

⇒ 3 odd days.

300 years give us 5 × 3 = 15 – 14 (two weeks)

⇒ 1 odd day.

400 years give us {5 × 4 + 1 (leap century)} – 21(three weeks)}

⇒ 0 odd days. Multiple of 400 years will gives you 0 odd days.

So odd days in the month of January = 31 – 28 = 3 odd days.

Similalry in the month of February(for non-leap year) 28 – 28 = 0 odd day and 29 – 28 = 1 odd day in a leap year and similarly for all the other months we can find the odd days contribution.

So we can sat that in first six months taken together i.e. January to June we get 6 odd days in a normal year (non-leap year) and 0 odd days in a leap year.

This is going to be of use, when the day we want to find is post 30th of June.

In the duration of first nine months which is from January to September we get 0 odd day for a (non-leap year) normal year and for a leap year it is going to be 1 odd day.

When the process of determining the days starts from the very beginning i.e. 1st January, 0001 then,

1 odd day ⇒ Monday

2 odd days ⇒ Tuesday

3 odd days ⇒ Wednesday

And so on..

Similarly, 6 odd days $\Rightarrow$ Saturday.

0 odd day $\Rightarrow$ Sunday.

Solved Examples:

1. On which day of the week does 2nd March, 1906 fall?

Ans. 2nd March,1906 = 1905 years + 1st Jan.1906 to 2nd March 1906
=1600 + 300 + 5 + 1st Jan.1906 to 2nd March 1906
The total count of 'odd days' in a duration of 1600 years = 0. The total count of 'odd days' in a duration of 300 years = 1. The total count of 'odd days' in a duration of 5 years (including a leap year) = 6

Month: Jan + Feb + Mar
Odd days: $3 + 0 + 2 = 5$
Total odd days $= 0 + 1 + 6 + 5 = 12$
Net odd days = when 12 divided by 7 remainder is 5 = 5 odd days. So 2nd March, 1906 falls on Friday.

2. If 6th march, 1998 was a Tuesday, then what will be the day on 2nd April 2001?

Ans. Odd days from 6th March 1998 – 6th March 1999 is 1 day
6th March 1999 – 6th March 2000
= 2 days (as 2000 is leap year)
6th March 2000 – 6th March 2001 = 1 day
Now from 6th March 2001 to 2nd April there will be $25 + 2 = 27$ which on dividing by 7 will give 6 as remainder.
So 6 more odd days.
So total odd days $1 + 2 + 1 + 6 = 10$.
when 10 is divided by 7 remainder is 3 = 3 odd days
So day on 2nd April 2001 will be Tuesday $+ 3 = 2 + 3 = 5$ means Friday.

15 Sets & Venn Diagram

1. Basic Operations on Sets

1. **Intersection of two sets:** The intersection of two sets is the set of elements common to both the given sets. The intersection of two sets P and Q is denoted as $P \cap Q$.
2. **Union of two sets:** The union of two sets is the set containing the elements belonging to P and also the elements belonging to Q. The union of these sets is denoted by $P \cup Q$.

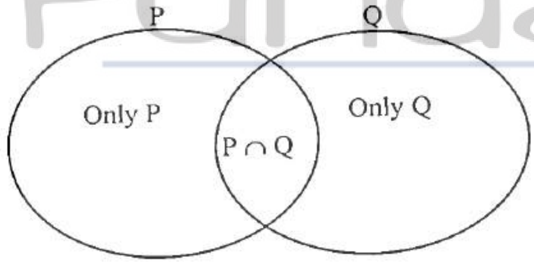
2. Venn Diagram of Two Objects

If the two given sets are 'P' & 'Q', then number of elements in union of P and Q is given by:

$n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

$n(P \cup Q)$ is also equal to n(Only P) + n(Only Q) + $n(P \cap Q)$ Where "$P \cap Q$"

$\Rightarrow$ "P intersection Q" is part which is common in P & Q.

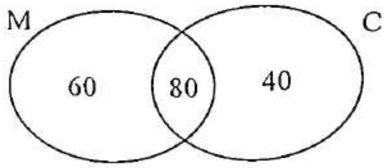


Solved Example:

1. In a college, 200 students are randomly selected. 140 like milk, 120 like coffee and 80 like both milk and coffee.

1. How many students like only milk?
2. How many students like only coffee?
3. How many students like neither milk nor coffee?
4. How many students like only one of milk or coffee?
5. How many students like at least one of the beverages?

Ans. The given information may be represented by the following Venn diagram, where M = milk and C = coffee.



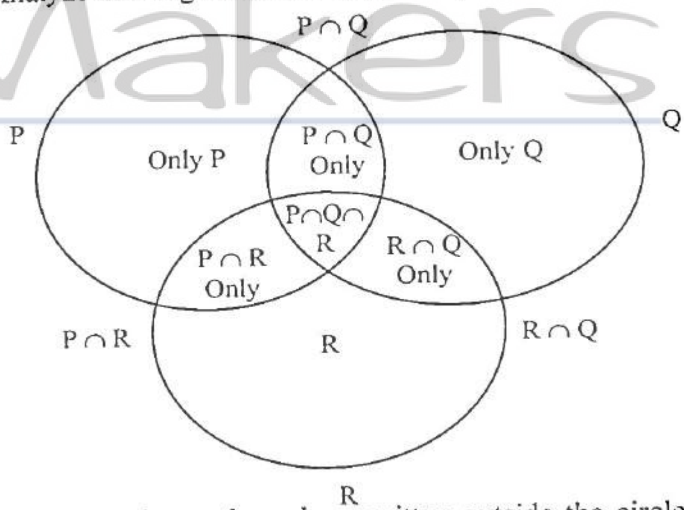
1. Number of students who like only milk = 60
2. Number of students who like only coffee = 40
3. Number of students who like neither milk nor coffee = 20
4. Number of students who like only one of milk or coffee = 60 + 40 = 100
5. Number of students who like at least one of milk or coffee = n (only Milk) + n (only coffee) + n (both Milk & coffee) = 60 + 40 + 80 = 180

3. Venn Diagram of 3 Objects

If the three given sets are 'P', 'Q' & 'R', then number of elements in union of P, Q and R is given by

$n(P \cup Q \cup R) = n(P) + n(Q) + n(R) - n(P \cap Q) - n(Q \cap R) - n(R \cap P) + n(P \cap Q \cap R)$

Analyze the diagram of the three components.

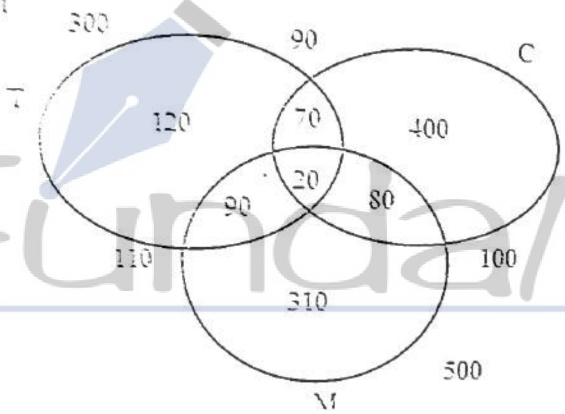


As shown above, the values written outside the circle is for whole object and those written inside the circle are exclusively for that region.

2. In a society everyone like at least one of the drinks from tea, coffee or milkshake. 300 like tea, 400 like coffee only, 500 like milkshake. 90 like tea and coffee, 100 like coffee and milkshake, 110 like milkshake and tea 20 like all three drinks. Now answer the following questions

1. How many people are there in society?
2. How many like only tea and only coffee?
3. How many like exactly one drink?
4. How many like exactly two drinks?

The whole information can be summarized in following Venn diagram



Ans.

1. Total number of people in society
   $= 120 + 310 + 400 + 70 + 90 + 80 + 20 = 1090.$
2. Total number who like only tea and only coffee
   $= 120 + 400 = 520.$
3. Total number who like exactly one drink
   $= 120 + 400 + 310 = 830.$
4. Total number who like exactly two drink
   $= 70 + 90 + 80 = 240.$

Probability

Probability is chances of happening of an event. Mathematically, it is the ratio of favourable outcomes to the total number of outcomes or

$$P(A) = \frac{\text{Favourable outcomes}}{\text{Total number of outcomes}}.$$

In probability, we generally come across the terms like experiments. Experiments are of two types: Deterministic and random experiments.

1. Definitions

Equally likely outcomes: The outcomes in a sample space S are equally likely if each outcome has the same probability of occurring. E.g., if a normal coin is tossed, then the chances of occurrence of head or tail are same. So, both the outcomes are equally likely.

Exhaustive number of cases: The total number of possible outcomes of a random experiment in a trial is called the exhaustive number of cases.

Mutually Exclusive Events: The events are said to be mutually exclusive if the occurrence of any one of them blocks the occurrence of all the others i.e., if all the events cannot occur simultaneously in the same trial. E.g., if we throw a coin, then occurrence of head or tail are mutually exclusive.

Independent Events: The events are said to be independent if the happening (or non-happening) of one event is not effected by the happening (or non–happening) of other. If two dice are thrown together, then getting an odd number on first is independent of getting an even number on the second.

Definition of Probability: If there are n–elementary events associated with a random experiment and m of them are favourable to an event A then probability of A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$.

Thus $P(A) = \frac{m}{n}$, since $0 \le m \le n$ therefore $0 \le \frac{m}{n} \le 1$, therefore $0 \le P(A) \le 1$. If $P(A) = 1$, A is called certain event and if $P(\bar{A}) = 1$ (or $P(A) = 0$), A is called impossible event.

If a coin is tossed 'n' times or 'n' coins are tossed only once, then the total number of outcomes = 2^n.

If a die is rolled 'n' times or 'n' dice are rolled once, then the total number of outcomes = 6^n.

2. Addition Theorem of Probability

If A and B are two events associated with a random experiment, then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are mutually exclusive events, then $P(A \cap B) = 0$. Therefore $P(A \cup B) = P(A) + P(B)$.

If A, B, C are three events associated with a random experiment, then

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$.

Solved examples:

1. Three coins are tossed together. Find the probability of getting:
   (a) At most two heads
   (b) Exactly two heads
   (c) No tails
   (d) At least one head and one tail

Ans. Total number of cases = $2^3 = 8$
(2^n, where n is number of coins)
Sample Space = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

(a) At most two heads = 1 – P(3 heads) $= 1 - \frac{1}{8} = \frac{7}{8}$

(b) Exactly two heads = {HHT, HTH, THH} = 3 cases

P(exactly 2 heads) $= \frac{3}{8}$

(c) No tails = all heads, that is only one case {HHH}

Required probability $= \frac{1}{8}$

(d) At least one head and one tail = {HHT, HTH, THH, TTH, THT, HTT} = 6 cases

Required probability $= \frac{6}{8} = \frac{3}{4}$

2. A die is thrown. Find the probability of getting:

(a) A multiple of 3 (b) Prime number
(c) A composite number (d) A number less than 5
(e) An even prime number

Ans. Total number of cases = 6{1, 2, 3, 4, 5, 6}

(a) Multiple of 3 = {3, 6} = 2 numbers

$$P(3k) = \frac{2}{6} = \frac{1}{3}$$

(b) Prime numbers = {2, 3, 5} = 3 numbers

P (prime number) $= \frac{3}{6} = \frac{1}{2}$

(c) Composite numbers = {4,6} = 2 numbers

P (composite number) $= \frac{2}{6} = \frac{1}{3}$

(d) Natural number less than 5
= {1, 2, 3, 4} = 4 numbers

Probability (number less than 5) $= \frac{4}{6} = \frac{2}{3}$

(e) Even prime number = {2} = 1 number

P (even prime number) $= \frac{1}{6}$

3. Two dice are thrown simultaneously. What is the probability of getting a sum of 9 or 10?

Ans. Total number of cases = $6^2 = 36$

Event (A) = where sum is 9
= {(6, 3) (3, 6) (5, 4) (4, 5)}, n(A) = 4
Event (B) = where sum is 10
= {(6, 4) (4, 6) (5, 5)}, n(B) = 3

P (A or B) $= \frac{4}{36} + \frac{3}{36} = \frac{7}{36}$

4. A card is drawn at random from a well shuffled pack of 52 playing cards. Find the probability of getting either a red card or a queen.

Ans. Total Cards = 52, Red cards = 26, Queens = 4
P (either a red card or a queen)

$$= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$$

5. If two different dice are rolled together, calculate the probability of getting an even number on both dice.

Ans. Total cases 6 × 6 = 36
Odd numbers on both the dice can be obtained as
= {(2,2), (2,4), (2,6), (4,2), (4,6), (6,2), (6,4),
(4,4), (6,6)}.

Favorable cases = 9

P (both odd numbers) $= \frac{9}{36} = \frac{1}{4}$

Data

1. Introduction

Before going into details, let us understand 'data'. Consider the marks obtained by 20 students of a class in a math test, where the maximum marks are 50. The scores of the 20 students are as follows

8, 2, 6, 17, 16, 2, 6, 18, 20, 15, 14, 8, 7, 9, 25, 5, 7, 8, 10, 12.

Now the marks obtained can be shown or collected in different ways for their analysis:

(a) In the illustration above, we have shown the marks of 20 students written in no definite pattern i.e., we have just asked the student and written down his/her marks. This method of collection and representation of data is known as ungrouped data.

(b) The other way of showing data is to see that how many times any particular marks are obtained by how many students.

For example in the above data:

Marks obtained	No. of Students
2	2
8	3
6	2

And so on.

In this method of collection and representation, we check how many times a particular observation (in this case, marks) occurs. This is known as discrete frequency distribution of the data or simply, the frequency of that observation.

(c) Another way is to divide the given data in small groups or intervals with each observation placed in the interval in which it lies.

Let intervals be 0-10, 10-20, 20-30, 30-40, 40-50 i.e., we have divided 50 marks into 5 intervals of width 10 each.

Now marks (2, 6, 7, 8, 9) will come under interval 0-10. Marks (10, 12, 14, 17, 18, 15, 16) will come under 10-20 and so on. In tabular form:

Class Interval	Frequency
0 – 10	11 (as there are marks of 11 students in this interval)
10 – 20	7 (as the marks of 7 students lies in the interval)
20 – 30	and so on.

This method is known as continuous frequency distribution method.

(d) Alternately, we could have designed the ranges in such a way that both upper and lower limits are included. (inclusive distribution)

Class Interval	Frequency
0 – 10	12 (as there are marks of 12 students in this interval)
11 – 20	7 (as the marks of 7 students lie in this interval)
21 – 30	and so on.

So we have seen a few ways to represent the collected data. After the data is collected it is analyzed as per requirements.

2. Definitions

Statistics - It is the science which deals with collection, presentation, analysis & interpretation of numerical data.

(a) **Class Interval:** The small group in which we place each observation is known as class interval. For example 0 – 10, 10 – 20 etc. are class intervals.

(b) **Class Frequency:** The number of observations which comes under any class interval is the frequency of that class interval. For example in the above marks distribution of the students, 11 is the class frequency of class interval 0 – 10, 7 is class frequency for class interval 10 – 20 and so on.

(c) **Limit:** Each interval has two limits. For example in class interval 0 – 10, 0 is lower limit and 10 is upper limit.

(d) **Class Marks or mid value:** It is the mid point of any class interval for example:

for class interval of 10 – 20 the class marks is $\frac{10+20}{2}=\frac{30}{2}=15$

(e) **Frequency:** It is a number representing how many times that observation occurs in the data.

(f) **Cumulative frequency:** The sum of preceding frequencies of class intervals.

(g) **Measures:** The terms we calculate and use for analysing data. They are broadly divided into 2 major categories or measures.

Measures of Central Tendency

In this category we have the terms:-

(a) Mean

(b) Median

(c) Mode

Note: When the values of mean or median or mode are calculated, it has been observed that these values lie within the data and are considered as true or standard values or very close to standard values. That's why mean, median or mode are known as measure of central tendency.

But when we calculate the values of range, deviation etc. of the data, it is observed that these values lie outside the range of data. That is why these parameters are known as measures of dispersion. In this measure we try to see how close or how far from true or standard values, our data is.

1. Mean

(1a) Arithmetic Mean or Mean or A.M.:

Arithmetic Mean is a numerical value obtained by dividing the total sum of all observations by total number of all observations.

So if x_1, x_2, x_3......... upto x_n are n observations then arithmetic mean of these n observations which is denoted by $\overline{x}$ is given by

$$\overline{x}=\frac{x_1+x_2+x_3+............+x_n}{n}$$

$$=\frac{1}{n}\sum_{i=1}^{n}x_i$$

where $\sum_{i=1}^{n}x_i$

(read as summation x_i) is a notation to represent the sum of all considered observations.

Note: This mean is for **ungrouped data.**

And if the data is **grouped** i.e. observations x_1, x_2, x_3........x_n have frequencies f_1, f_2................ f_n respectively then the mean is given by

$$\overline{x}=\frac{x_1f_1+x_2f_2+............+x_nf_n}{f_1+f_2+f_3+.........+f_n}$$

$$=\frac{\sum_{i=1}^{n}x_if_i}{\sum_{i=1}^{n}f_i}=\frac{\sum_{i=1}^{n}x_if_i}{N}, i=1,2,3......n$$

where $\sum_{i=1}^{n}f_i=N$, is total number of observations.

(1b) Change of Origin Method for Calculating Mean:

Let A be any assumed number (usually taken in the middle of the series) and d_i be the deviation about A i.e., $d_i=x_i-A$.

The Arithmetic mean $\overline{x}$ is given by

$$\overline{x}=\frac{\sum f_i x_i}{N} \quad [\because d_i=x_i-A]$$

$$=\frac{1}{N}\sum f_i(A+d_i)=\frac{1}{N}\sum f_i\cdot A+\frac{1}{N}\sum f_i d_i$$

$$=A+\frac{1}{N}\sum f_i d_i \qquad \left(\because \sum f_i=N\right)$$

Here A is called Assumed Mean and d_i is called deviation about A.

(1c) Group mean or combined mean:

If two groups with mean $\overline{x_1}$ and $\overline{x_2}$ have n_1 and n_2 number of observations respectively, then the combined mean of the group is given by $\overline{x}=\frac{n_1\overline{x_1}+n_2\overline{x_2}}{n_1+n_2}$

(1d) Weighted Mean:

When the weights of each observation instead of frequency are given, then the mean of this set of observation is known as weighted mean.

$$\overline{W}=\frac{w_1x_1+w_2x_2+w_3x_3+.....+w_nx_n}{w_1+w_2+w_3+.....+w_n}$$

where w_1, w_2, w_3, ... w_n are weights of observations x_1, x_2, x_3, ... x_n respectively.

For the sake of understanding, consider weights, as **importance** in terms of number, given to that observation.

Some facts about arithmetic mean:

(a) If every observation is increased or decreased by same number, then the arithmetic mean of new observations also increases or decreases by same number.

(b) If every observation is multiplied or divided by same number, then arithmetic mean of the new set of observations is obtained by multiplying or dividing the initial arithmetic mean by same number.

Solved examples related to Mean:

1. Find the mean of data given below
(i) 15, 20, 25, 10, 5, 30, 12, 8
(ii) If the frequencies of observation of above data are 2, 4, 3, 5, 7, 6, 1, 8 respectively.

Ans. i) here total observations are $n = 8$ and x_1, x_2, x_3...... are 15, 20, 25, 10, 5.........

$$\therefore \text{ mean } = \frac{15+20+25+10+5+30+12+8}{8} = 15.62$$

(ii)

x_i	f_i	f_ix_i
15	2	30
20	4	80
25	3	75
10	5	50
5	7	35
30	6	180
12	1	12
8	8	64
	$\sum f_i = 36$	$\sum f_ix_i = 526$

We know when frequency is given then mean

$$= \frac{\sum f_ix_i}{\sum f_i} = \frac{526}{36}$$

$$= 14.611$$

2. The average marks of 80 students was found to be 50. Later it was disclosed that a score of 35 was misread as 65. Find the correct mean of 80 students.

Ans. We know that $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n}$ ...(i)

Here $\bar{x} = 50$, n = 80.

$\therefore$ from (i) $\sum_{i=1}^{80} x_i = 80 \times 50 = 4,000$ i.e.,

total marks of whole class is 4,000

Now we will deduct the score of 65 from the total marks and add the score of 35 i.e.,

$(4000 - 65) + 35 = 3970$

$\therefore$ Correct mean $= \dfrac{3970}{80} = 49.62$ (approx.)

2. Median

It is size or the value of observation which lies in the middle of the observations. This mid-point or median depends upon whether the total number of observations are odd or even.

(2a) Ungrouped Data (frequency not given):

First arrange the data in ascending or descending order and if total observations are n then we have:

If n is odd, median = value of $\left(\frac{n+1}{2}\right)^{th}$ observation and if n is even then median $= \frac{1}{2}$ [value of $\left(\frac{n}{2}\right)^{th}$ item + value of $\left(\frac{n}{2}+1\right)^{th}$ item]

(2b) For discrete series (frequency is given):

First arrange the data in ascending or descending order, then we calculate the cumulative frequencies of the observations. Median = $\left(\frac{n+1}{2}\right)$th observation where n is $\sum f_i$.

(2c) For grouped or continuous frequency distribution.

If series is in ascending order

Then median $= l + \dfrac{\left(\frac{n}{2} - c\right)}{f} \times i$, where l = lower limit of the median class, n = sum of all the frequencies.
i = width of median class, f = frequency of median class.
c = cumulative frequency of class preceding median class.

Median class is the interval where the value of $\frac{n}{2}$ th lies.

Solved example based on Median:

1. Find the median of data given below
(i) 6, 11, 13, 15, 17, 21, 24
(ii) x: 10, 15, 20, 25, 30, 35
f: 7, 18, 19, 6, 8, 24
(iii) class interval: 0-10, 10-20, 20-30, 30-40, 40-50
frequency : 4 8 11 7 8

Ans. (i) Here total observations are 7, which are odd in number. So to locate the median, the data is arranged either in ascending order or descending order but here data is already given in ascending order i.e., 6, 11, 13, 15, 17, 21, 24.

$\therefore$ median = value of $\left(\frac{n+1}{2}\right)^{th}$ item value of $\left(\frac{7+1}{2}\right)^{th}$ item = value of 4th item = 15

(ii)

x	f	$c.f$
10	7	7
15	18	25
20	19	44
25	6	50
30	8	58
35	24	82

now value of $\left(\frac{n+1}{2}\right)^{th}$ item $= \left(\frac{82+1}{2}\right)^{th}$

$$= \left(\frac{83}{2}\right)^{th} = 41.5$$

The value of $\left(\frac{n+1}{2}\right)^{th}$ item (which is 41.5) falls in 3rd observation in cumulative frequency table which corresponds to 3rd observation in value of variables ($x_1, x_2, \ldots\ldots\ldots\ldots$)

$\therefore$ median of this data = 20

(iii)

$C.I$	*Frequency*	$C.F$
0-10	4	4
10-20	8	12
20-30	11	23
30-40	7	30
40-50	9	39
	N = 39	

$$\text{median} = l + \frac{\left(\frac{N}{2} - c\right)}{f} \times i$$

Now $\frac{N}{2} = \frac{39}{2} = 19.5$, which corresponds to class interval 20-30 as per cumulative frequency table

$\therefore$ 20-30 is median class

$$\therefore \text{Median} = 20 + \left(\frac{19.5-12}{11}\right) \times 10 = 26.818$$

3. Mode

The value of the observation(s) which occurs maximum number of times in the data is/are called its mode. Generally we locate this value just by inspecting the data.

(3a) For individual series or raw data:

The mode is that observation(s) which is occurring maximum number of times.

(3b) For discrete data (with frequency):

The mode is observation(s) having highest frequency.

(3c) For grouped data with continuous frequency distribution.

$$\text{Mode} = l_1 + \left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right) \times i$$

Here we first locate the modal class, which is class interval with highest frequency. Then

l_1 = lower limit of modal class.

f_m = frequency of modal class,

f_1 = frequency of the class preceding to modal class.

f_2 = frequency of class succeeding to modal class,

i = size of class internal of modal class.

Relation between Mean, Median and Mode:::
Mode = 3 Median – 2 Mean.

Solved examples based on Mode:

1. Find the mode of data given below
   (i) 2, 2, 3, 5, 7, 8, 9, 2, 2, 9
   (ii) x : 10, 15, 20, 25, 30, 35
   f : 7, 18, 16, 14, 12, 10
   (iii) Class interval: 0-10, 10-20, 20-30, 30-40, 40-50
   Frequency : 4 8 11 7 8

Ans. (i) We know mode is that observation(s) in data which appears maximum no. of times in that data. Here 2 is coming four times in data, which is maximum in occurrence.

$\therefore$ mode of this data is 2.

(ii) In this data frequency of observation 15 is maximum (18)

$\therefore$ mode of this data is 15.

(iii) Here frequency of C.I, 20-30 is maximum

$\therefore$ 20-30 is modal class

$$\text{Mode} = l_1 + \left(\frac{f_m - f_1}{2f_m - f_1 - f_2}\right) \times i,$$

where l_1 = lower limit of modal class, here $l_1 = 20$

f_m = frequency of modal class, here $f_m = 11$,

f_1 = frequency of class preceding to modal class here $f_1 = 8$

f_2 = frequency of class succeeding to modal class here $f_2 = 7$, i = width of class interval = 10.

$$\text{Mode} = 20 + \left(\frac{11-8}{22-8-7}\right) \times 10$$

$$= 20 + \left[\frac{3}{7}\right] \times 10$$

$$= 24.28$$

2. In a certain distribution, the mean and median are 26 and 24 respectively. Find the value of the mode.

Ans. We know that mode = 3 median – 2 mean

$\therefore$ mode = 3(24) – 2(26)
= 72 – 52
= 20

Some important Definitions:

1. The geometric mean is the n^{th} root of the product of n individual observations.
2. The harmonic mean is the reciprocal of the arithmetic mean of the reciprocals of the individual observations.
3. **Range:** It is the difference in values of highest and lowest observations in the given data.
   Range = highest observation – lowest observation.